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SPECIAL MONOGRAPH ISSUE

STUDIES OF TARGET DETECTION BY PULSED RADAR

BY

J. I. MARCUM AND P. SWERLING

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INFORMATION THEORY

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Special Monograph Issue

STUDIES OF TARGET DETECTION BY PULSED RADAR

With this issue the IRE Professional Group on Information Theory begins a new service to its members — the publication of a series of Monographs. These volumes will appear in the TRANSACTIONS series and will supplement the usual four issues per year. They will appear at irregular intervals as our finances permit and as important contributions become available that are too lengthy for the regular TRANSACTIONS. The decision to undertake this venture is based principally on the conviction that our profession has lacked in the past a suitable vehicle for making available at low cost the class of papers that are too long for the TRANSACTIONS, but have not received and are not likely to receive commercial publication as books.

We would like to express our appreciation to the RAND Corporation, a contractor to the U.S. Air Force, for their kind permission to reprint the Marcum and Swerling reports as our first Monograph.

— *The Editors.*

A Statistical Theory of Target Detection

By Pulsed Radar

(RAND Research Memo. RM-754, December 1, 1947)

by

J.I. Marcum

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LIST OF FIGURES

ERRATA

In the first footnote on page 79, η should be replaced by n.

On page 81, in (24),

$$\int_{R_1}^{R} (1-P)\, dR \quad \text{should be} \quad \int_{R_1}^{R} \log_e (1-P)\, dR$$

The same correction should be made in (25).

In Figs. 1 thru 44 and Fig. 53, all of the ordinates appear as percentages but are labeled as probabilities. Therefore, in order to make the scales conform, the decimal place should be moved two units to the left on all of the numbered ordinates. For instance, 1 would become .01, etc.

SUMMARY

This report presents data from which one may obtain the probability that a pulsed-type radar system will detect a given target at any range. This is in contrast to the usual method of obtaining radar range as a single number, which can be taken mathematically to imply that the probability of detection is zero at any range greater than this number, and certainty at any range less than this number.

Three variables, which have so far received little quantitative attention in the subject of radar range, are introduced in the theory:

1. The time taken to detect the target.
2. The average time interval between false alarms (false indications of targets).
3. The number of pulses integrated.

It is shown briefly how the results for pulsed-type systems may be applied in the case of continuous-wave systems.

Those concerned with systems analysis problems including radar performance may profitably use this work as one link in a chain involving several probabilities. For instance, the probability that a given aircraft will be detected at least once while flying any given path through a specified model radar network may be calculated using the data presented here as a basis, provided that additional probability data on such things as outage time etc., are available.

The theory developed here does not take account of interference such as clutter or man-made static, but assumes only random noise at the receiver input. Also, an ideal type of electronic integrator and detector are assumed. Thus the results are the best that can be obtained under ideal conditions. It is not too difficult, however, to make reasonable assumptions which will permit application of the results to the currently available types of radar.

The first part of this report is a restatement of known radar fundamentals and supplies continuity and background for what follows.

The mathematical part of the theory is not contained herein, but will be issued subsequently as a separate report[23].

SYMBOLS

A_e = effective area of antenna for receiving.
B = beamwidth of antenna.
c = velocity of light.
C = total shunt capacity of input circuit.
δ = factor which accounts for losses in transmission lines, T R switches, atmospheric absorption, etc.
e = rms noise voltage.
E_p = transmitted energy per pulse.
E_R = received energy per pulse.
f_r = pulse repetition frequency.
f_{sc} = scanning frequency.
Δf = bandwidth for noise purposes.
$\Delta f'$ = input circuit bandwidth.
Δf_{cw} = combined R F and I F bandwidth of continuous-wave-system receiver.
F = bandwidth multiplying factor = 1 for simple L C circuit.
γ = number of pulses received during detection time.
γ' = γ/N
g_m = mutual conductance of first receiver tube.
G = gain of transmitting antenna.
h_r = height of radar antenna.
h_t = target height.
$I_0(z)$ = modified Bessel function of the first kind.
k = Boltzmann's constant.
λ = wave length of transmitter.
L = sweep length in miles.
n = $\tau_{fa} \cdot f_r \cdot \eta$
n' = n/N
η = number of pulse intervals per sweep.
η' = number of separate velocity channels in continuous-wave-system receiver.
N = number of pulses integrated, or, in cw system, the number of independent variates (of length $1/\Delta f_{cw}$) integrated.
N_{sc} = number of pulses per scan.
$\overline{NF}$ = overall noise figure of the receiver.

P_N = probability that N pulses of noise will exceed a given level.

P = probability that N pulses of signal plus noise will exceed the bias level.

P' = probability that at least one group of N integrated pulses will exceed the bias level within the detection time.

P_{av} = average power.

P_t = transmitted power.

P_{min} = minimum detectable power at receiver.

p = effective input noise power to receiver.

r = resistance.

R = radar range.

R_{max} = maximum radar range.

R_0 = idealized radar range.

R_{eq} = equivalent noise resistance of first receiver stage.

R_1 = total shunt resistance of first receiver input circuit.

ΔR = range interval for integration with a moving target.

σ = scattering cross-sectional area of target.

τ_p = pulse length.

τ_{fa} = false alarm interval.

τ_d = detection time.

τ_i = maximum integration time for moving target.

T = absolute temperature.

T_a = absolute temperature of space radiation received by antenna.

T_R = absolute temperature of room.

v = velocity of the target.

v_g = velocity of traveling gate.

V = visability factor of receiver.

ω = angular velocity of antenna.

x = received signal pulse energy in units of $k\ T_R\ \overline{NF}$.

y = noise level in units of the rms value of noise — the bias level.

A STATISTICAL THEORY OF TARGET DETECTION BY PULSED RADAR

PART I - INTRODUCTION

THE USUAL RADAR RANGE EQUATION

Most radar engineers are now well acquainted with the following equation used to determine the maximum range of a pulsed radar system:

$$R_{max} = \left[\frac{P_t}{P_{min}} \frac{GA_e \sigma \delta}{16 \pi^2}\right]^{1/4} \tag{1}$$

where

P_t = peak transmitted power in watts.

P_{min} = minimum peak detectable signal in watts.

σ = scattering cross section of target in units consistent with range.

G = gain of transmitting antenna.

A_e = effective area of antenna for receiving in units consistent with range (usually about 2/3 of the physical aperture, $A_e = G\lambda^2/4\pi$).

δ = a dimensionless loss factor which accounts for atmospheric absorption, losses in antenna and transmission lines, etc.

The number of pitfalls that may be encountered in the use of the above equation are almost without limit, and many of these difficulties have been recognized in the past[(3), (18)]. Three of the most troublesome are:

1. *The Scattering Cross Section*

   In the case of moving targets, the wide variation of this quantity with aspect, and hence with time, is a matter of vital concern. The variation of cross section as a function of frequency may also be critical.

2. *The Minimum Detectable Signal,* P_{min}

   The statistical nature of the noise with which P_{min} must compete makes this an ill-defined quantity.

For references see page 143.

3. The Maximum Range, R_{max}

The statistical nature of P_{min} in turn makes R_{max} a statistical quantity.

There are also lesser troubles, such as the dependence of δ, the loss constant on the range, and the contribution of reflections from the ground, sea, or other objects to the incident and received powers. (One must also remember that a target cannot ordinarily be detected at ranges (in miles) much greater than $\sqrt{2h_r} + \sqrt{2h_t}$, where h_r is the height of the radar antenna and h_t the height of the target in feet, except in the case of superrefraction, or "ducts." See pp. 55-58, Ref.(18). This is the familiar "line of sight" limitation due to the earth's curvature.

THE SCATTERING CROSS SECTION OF THE TARGET

For a stationary radar observing a stationary target, the scattering cross section is a constant. Although it may not be calculated for any but the most simple target shapes, it is not too difficult to measure. On the other hand, if either the radar or the target is in motion, the cross section becomes a function of time causing the return signal strength to fluctuate. In general, the plot of cross section as a function of angle for a complex target such as an aircraft shows two interesting features. There is a nearly continuous rapid fluctuation having an angular period in the neighborhood of a degree or so (for λ in the microwave region), and a slow variation with a period in the order of 20° or more. Both of these variations may be as great as 30 db. The question at once arises: In lieu of using the complete polar diagram of cross section vs. angle, what kind of average figure can be used, and under what conditions? The answer to this question involves such things as angular rates of the aircraft with respect to the radar, correlation times, repetition rate of the radar, and number of pulses integrated. It is almost obvious that the only general way to treat this complex problem is to consider the cross section as a statistical variable. This approach seems mathematically feasible. However, in the present report the cross section will be considered to be a constant. An attempt to justify this assumption is the following: The rapidly fluctuating correlation angle at half-power points is perhaps 0.1°. The normal variation in attitude angle of an aircraft may be about 30° per second. (This variation may be caused by small rapid changes in pitch or roll due to normal turbulence of the air as well as by systematic changes in position.) Thus, the corresponding correlation time for σ is around 1/300 second. If the observation time is essentially greater than this period, it may be assumed, as a first approximation, that the rapid fluctuations in the cross section "average out."

The slow variations (period around 20°) may or may not average out. However, if the average over all *likely* attitudes is used for σ, or to be more exact, if a weighted average is taken for σ according to the probability for any attitude, then the probability of detection may not be changed very much. Henceforth, in this report σ will be assumed to be a constant, on the basis of the above statements. It may be mentioned in passing that σ loses its meaning if the target is not uniformly illuminated. Such can be the case, for example, if waves reaching the target via two or more paths combine to produce an interference pattern at the target. This effect exists in the detection of ships by surface radar.

THE MINIMUM DETECTABLE SIGNAL

As is well known,[1,2,3] the minimum detectable signal power in a radar receiver is fundamentally limited by three main factors; i.e., Johnson noise in circuit elements of the input circuits, shot effect and other noise in the first tube (and to some small extent succeeding tubes), and cosmic noise picked up by the antenna. There may also be man-made interference such as engine noise, radiations from other radars and radio transmitters, etc. Clutter caused by sea return, rain, clouds, land masses, etc., may reduce the minimum detectable signal by a considerable amount. The effects of clutter and man-made interference are complete subjects in themselves,[19] and will not be treated further in this paper. A study will be made here of radar range in the absence of such interference. It is not too optimistic to suppose that circuits will eventually be designed which will largely eliminate man-made interference, and most types of clutter.

The mean squared noise voltage across a resistor of resistance r is given by

$$e^2 = 4kTr\,\Delta f \tag{2}$$

where

k = Boltzmann's constant = 1.38×10^{-23} joules/degree
T = absolute temperature of the resistor
Δf = the frequency interval under consideration.

Though the noise at the input circuit of a receiver is usually several times this value, it provides a convenient scale for measuring the input noise. The effective input noise power is defined to be

$$p = kT_R\,\Delta f\,\overline{NF} \tag{3}$$

where $\overline{NF}$ is the so-called noise figure of the receiver, and T_R is the absolute *room* temperature.* If a signal power of the same value as p were *incident* on the antenna and the receiver were noiseless, then the output would be the same as in the case when noise only was present.

At this point, one important result concerning the noise figure due to Herold[1] is pertinent:

$$\overline{NF} = \frac{T_a}{T_R} + \frac{2\pi\Delta f' R_{eq} C}{F} + \underbrace{f(R_1)}_{\to 0 \text{ as } R_1 \to \infty} \tag{4}$$

where

T_a = absolute temperature of space radiation received by the antenna.
T_R = room temperature.

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* Complete discussions and derivations will be found in the Mathematical Appendix (a separate report).[23]

$\Delta f'$ = bandwidth of input circuit.

C = total shunt capacity of input circuit.

R_{eq} = equivalent noise resistance at input (due mainly to shot noise in first tube) $\approx 2.5/g_m$ for triodes.

F = a factor depending on the exact type of input circuit coupling (= 1 for simple tuned circuit).

R_1 = input shunt resistance including effect of finite input resistance of tube.

$f(R_1)$ = a function of R_1, R_{eq}, C and $\Delta f'$.

This formula assumes a more or less conventional type of input tubes, such as the V H F triodes and pentodes. However, it seems reasonable to believe that the general conclusions which are reached from Eq.(4) will apply to velocity-modulated input tubes as well.

The main points to be noted about Eq.(4) are these:

1. $f(R_1)$ approaches zero as R_1 approaches infinity. R_1 may be increased by better tube design.

2. C/g_m should be made as small as possible in a tube used as the first amplifier.

3. Long pulses tend to allow smaller bandwidths for the input circuit, and hence lower noise figures.

4. If $R_{eq}\, C\Delta f'$ is made small enough, and R_1 large enough, the noise figure will approach T_a/T_R.

Point 4 is of the *greatest importance.* It sets a limit on the noise figure when there are no sources of noise in the receiver itself. Though such a receiver will never be built in practice, it may be possible some day to approach closely this ideal state. Then the input noise will be almost *entirely* dependent on the temperature of space;* or, in other words, on the noise received by the antenna from without the radar set. That this state of affairs is not yet at hand is evidenced by the fact that at present the noise figure for microwave receivers is around 10, and for longer-wave receivers perhaps as low as 3 or 4.

The concept, often stated, that the ideal noise figure of a receiver is 1.0 is erroneous.** This would be true only if the temperature of space were the same as room temperature. Actually the temperature of space decreases rapidly with decreasing wave length.[3]

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* Though the noise figure can be decreased by increasing T_R, this would increase the actual input noise, as is apparent from Eq. (3).

** The noise figure of a receiver may be defined in such a way that the antenna must be replaced by a resistor at room temperature equal to the radiation resistance of the antenna. In this case the ideal noise figure of the receiver would be 1.0.

The average space temperature* is around room temperature at 180 megacycles and drops to around 30° absolute at 450 megacycles.[12] No good measurements are available in the microwave region, but there is reason to believe that values of 10° or lower may be found. If this proves to be true, then it is conceivable that the noise figure of future microwave receivers may be improved by a factor of 100, which would mean that the range of radar sets could be more than tripled as a consequence of this one factor. It is certainly a field where research should be pushed to the utmost.

It has often been the practice to calculate the maximum range of a radar set from (1) by assuming that $P_{min} = kT_R \Delta f \overline{NF}$, or that the minimum detectable signal power is just equal to the average noise power.

This gives

$$R_{max} = \left[\frac{P_t G A_e \sigma \delta}{16\pi^2 k T_R \Delta f \overline{NF}} \right]^{1/4} \tag{5}$$

Now the energy per pulse is represented by

$$E_p = P_t \tau_p \tag{6}$$

where τ_p is the pulse length. Making this substitution in (5) gives

$$R_{max} = \left[\frac{E_p G A_e \sigma \delta}{16\pi^2 k T_R \overline{NF}} \right]^{1/4} \cdot \frac{1}{(\tau_p \Delta f)^{1/4}} \tag{7}$$

It is usually said that if $\tau_p \Delta f$ is made equal to 1, the amplitude of the pulse after passing through the amplifier will not differ much from the amplitude which would result if the pulse were infinitely long. Without further ado, $\tau_p \Delta f$ is put equal to 1, and the resultant equation

$$R_{max} = \left[\frac{E_p G A_e \sigma \delta}{16\pi^2 k T_R \overline{NF}} \right]^{1/4} \tag{8}$$

emerges as the radar range equation. Now the unfortunate fact (in some respects) is that the range of a radar set calculated by means of this formula often turns out to be rather close to the experimental range. Naturally, under these circumstances great effort has not been expended in investigating the validity of radar-range equations.

* There is a variation of the space temperature with direction[12]. When the antenna points near the horizon, the temperature may be higher than when it is pointed at the zenith. In particular, if any appreciable part of the radiation strikes the ground, the thermal radiation received from those directions will have a temperature nearly equal to the actual temperature of the surroundings.

The reasons for the agreement of equation (8) with experiment are many. First of all, the cross section has been, in most cases, determined by observing the maximum range of a particular target and solving equation (8) for σ. This one fact alone accounts in no small way for the agreement. Secondly, the fourth power law makes R_{max} rather insensitive to changes in the various parameters concerned in equation (8). A much fairer test is to compare respective values of R^4_{max} rather than R_{max}.

Equation (8) is in no sense perfect with regard to its agreement with experiment. Errors of as much as ± 30% are common, and factors of 2 can often be found. However, considering all the unknown factors present in an experimental determination of maximum range with an operational radar set, this agreement is considered to be quite good.

In any field of science, theoretical equations are deduced to explain observed data. However, one is very cautious in using these equations to predict results for other experiments where the values of many of the variables differ greatly from those used in the particular experiments already performed. Most of the radar sets built to date have operated within essentially narrow limits as far as some of the parameters are concerned. Particular examples are pulse repetition frequency, and, *most important*, the number of pulses integrated. This latter quantity is not even mentioned in equation (8); but, as will be seen in the next section, it is of vital importance.

The task is now two-fold:

1. To make a satisfactory statistical definition of the range of a radar system.

2. To determine the dependence of this quantity on the parameters of a (pulsed) radar system.

PART II

THE STATISTICAL PROBLEM OF THE MINIMUM DETECTABLE SIGNAL AND THE MAXIMUM RANGE

GENERAL BACKGROUND

It has been realized by many workers in the field* that the range of a radar set is a statistical variable and must be stated in terms of probabilities rather than in the exact terms of an equation such as (8). However, the evolution of a practical working theory does not seem to have been accomplished so far. The following work is a first step in that direction.

Before beginning the explanation of equations and derivations, it will be well to glance at some of the new ideas which will be included.

The random noise, which limits the range, can at intervals assume large values due to its statistical nature. This means that there will inevitably be times when a random fluctuation of the noise will be mistaken for a signal. The average interval at which such undesirable events take place will be called the false alarm time, and it will be found that the probability of detecting a target will be a function of this time. Let the reader at once be cautioned against thinking, "If it were a noise flash, I can easily tell by looking a little later. If it were a signal, it will still be there; if it were noise, it will be gone."

The second new parameter which will be introduced is the detection time. It is apparent that if an observer can spend sufficient time in deciding whether or not a target is present on an oscilloscope screen, the probability of a correct decision being reached will be increased. It is also obvious that in any practical situation in which radar is used one cannot take unlimited time to decide whether or not a target is present. To put things on a quantitative basis, the time in which a decision shall be rendered must be specified. In this event, there will not always be time for the "second look" just mentioned; but should time permit, then the probability of detecting a target will be increased at the expense of a longer detection time. Even so, there will still be a *certain lesser probability* that the noise flashes will occur on *both occasions*. Further, it will be found that the velocity of a moving target has an appreciable effect on the detection probability, due to the fact that the signal from such a target does not "remain stationary" (see page 80).

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* For an excellent qualitative statement of the problem, see Radiation Laboratory Series No.1. pp.35-47, Ref.(18).

It is desirable to present data in the most compact form, and the first step in this direction is the elimination of the necessity for the appearance of such parameters as E_p, G, A_e, σ, δ, and $\overline{NF}$ in the final results. To this end, a parameter R_0 is defined which is given by a slight modification of Eq. (7), as follows:

$$R = \left[\frac{E_p G A_e \sigma \delta V}{16\pi^2 E_R}\right]^{1/4} \qquad R_0 = \left[\frac{E_p G A_e \sigma \delta V}{16\pi^2 k T_R \overline{NF}}\right]^{1/4} \tag{9}$$

Here, the factor $1/\tau_p \Delta f$ has been replaced by V, the so-called visibility factor.* This factor will always be less than 1 but usually not less than 0.8, except when the Doppler effect is very large. R_0 will be called the "idealized range" for lack of a better term.

Now let the received energy per pulse at any range R be E_R. Then it is clear from the equations (9) that

$$\frac{R}{R_0} = \left[\frac{k T_R \overline{NF}}{E_R}\right]^{1/4} \tag{10}$$

and defining

$$x = \frac{E_R}{k T_R \overline{NF}} \tag{11}$$

gives from (10)

$$\frac{R}{R_0} = \frac{1}{x^{1/4}} \tag{12}$$

---

* The derivation of exact formulas and numerous curves of visibility factors as a function of pulse width, bandwidth, type of amplifier, and off-resonance of carrier frequency will be found in the Mathematical Appendix (a separate report) [23]. The visibility factor is actually given by

$$V = \frac{\left(\dfrac{E_{max}}{E_{ss}}\right)^2}{\tau_p \Delta f}$$

where E_{max} is the maximum voltage to which the pulse rises at the receiver output, and E_{ss} is the steady state voltage at the same point. The quantity $\left(\dfrac{E_{max}}{E_{ss}}\right)^2$ should be contained in (7) and (8) but is usually omitted because it is so near to unity when $\tau\Delta f = 1$. In the case where the bandpass characteristic of the amplifier is the conjugate transform of the pulse, the visibility factor is exactly unity [4].

where x is now the signal pulse energy in units of the average receiver noise pulse energy. As an example, suppose $x=4$, which means that the signal power equals four times the average noise power. Suppose the probability is calculated to be 0.5 that in this case the signal will be detectable. There is then a point $P=0.5$ at $R=0.7R_0$. When a series of such points are calculated for various values of x, a curve for P as a function of R/R_0 may be drawn, assuming fixed false alarm time, etc.

INTEGRATION OF PULSES

Before proceeding further, the meaning of pulse integration must be defined in detail. In its simplest form, it merely consists of adding N successive signal pulses together and attempting to detect the sum rather than an individual pulse. Now, whatever the integrating device may be, it will not know in advance whether there is a signal or not, and hence in the absence of a signal it will add up N successive noise pulses. Therefore, the comparison is between N signal plus noise pulses and N noise pulses as contrasted to a single signal pulse to a single noise pulse. One might be tempted to say that the signal to noise *ratio* would be unchanged, and that integration, or addition, of pulses therefore offered no advantage. This argument neglects the fact that the noise voltage fluctuates about its average value. The mean or average value of the noise voltage is not of too much concern, for it can always be "biased out."* If we add N signal pulses of voltage V, the total signal voltage is NV. If we add N noise pulses of average voltage V_N, the *average* sum will be NV_N. However, the average sum can be balanced out. The question is, whether or not the fluctuation in the sum voltage is now N times the fluctuation voltage of single pulse. If the answer were yes, then integration would be futile. However, due to the random nature of the fluctuation of any single pulse, the fluctuation voltage of the sum is only about $\sqrt{N}$ times the fluctuation voltage of a single pulse. *It is the signal to noise-fluctuation** ratio, not the signal to average noise ratio that is of paramount importance.* The greater the number of pulses integrated, the greater is the signal to fluctuation ratio, and the greater is the probability of detecting the signal, *but at the expense of longer detection times.*

DEFINITION OF DETECTION AND THE BIAS LEVEL

Before the false alarm time can be calculated, a definition of "detection of a signal" must be given. Detection of a signal is said to occur whenever the output of the receiver exceeds a certain predetermined value hereafter called the bias level. In the absence of any signal, this bias level will on occasion be exceeded by the noise alone. The higher the bias level is set, the more infrequently this happens. The first problem is to calculate the required bias level, given the false alarm time. Knowing this bias level, the rest of the problem is to calculate the probability that any given value of signal (plus noise) will exceed this level.

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* Practically, the bias level should not be too large, or the fluctuations in the bias will become of concern. See page 75 where a method of reducing the necessary bias level by a considerable factor is discussed.

** The mathematical term for the fluctuation is the "standard deviation," usually denoted by σ.

This is well and good, one says, but is this the best means of detection? What about the operator watching a cathode ray tube - what are his criteria for calling "signal"? Of course, it is impossible to say exactly, as is evidenced by the wide variation among radar operators. One can see, though, how an operator is affected by the false alarm time. If he is told that he will be subject to severe penalties if he calls a false alarm (calls a signal when it subsequently turns out that there was none), then he will be very cautious. If a doubtful pip appears on the screen, he will use discretion and say nothing. This means that under these conditions the false alarm time is increased, and at the same time the probability of detecting a target at a given range is decreased.

The operator may use the shape of a signal pulse contrasted to that of the noise as a criteria for detection as well as amplitude differences. This is thought to be a second order effect. The operator, on the other hand, is limited to some extent by the minimum brightness ratio which the eye can detect.

It seems that the method of electronic detection proposed above will be practically as good as any other possible method, electronic, human, or otherwise, if identical false alarm times and detection times are assumed. This statement is certainly not to be considered obvious. It should be possible to make some experiments to verify this theory.

METHODS OF PULSE INTEGRATION

As stated before, to integrate pulses it is merely necessary to add them together. There are many different practical ways in which this is done. One of the simplest is the use of a cathode ray tube screen[9]. Due to the screen persistence time, a certain number of pulses will be effectively integrated. In this case it will not be a simple addition, being more in the nature of a weighted average. The effect of weighting is always bad. In other words, the effect of equal samples in the integrated result should be as nearly the same as possible. P P I type of presentations which use intensity modulated displays usually have much longer integration times than an *A* scope.

One must not overlook the human operator,* who goes along with the cathode ray tube, as a vital part of the detection mechanism. The combination of the eye and the brain makes a very good integrator. In fact, the maximum integration time for a skilled operator may easily be several seconds. The best electronic integrators for pulsed radar built to date will not better this figure to any great extent. Henceforth, a model electronic integrator which linearly adds N pulses will be assumed.

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* There are a large number of factors involving observers and oscilloscopes which are quite complicated and are more or less outside the intended scope of this report. Lawson and his group have done a great deal of work on this subject, the results of which will appear in Chap.VIII of Ref.(19). Most of these experimental results are also available in Ref.(24).

Now, pulses can be integrated in the R F stages, in the I F stages, or in the video stages[(13), (14), (16), (19)]. Futhermore, there can be one or more linear or square law detectors present*, and the integration can be done in one or more steps and in at least two different ways. Many of these possibilities are reserved for detailed treatment in a separate mathematical report[(23)].

Fortunately, the results for the various cases show little difference, with one marked exception. R F and I F integration are better than video integration for small signals (compared to the noise). However, there is no practical way known at present to take full advantage of R F or I F integration with moving targets because of the requirement that the successive received pulses must be completely coherent[(13), (14)]. Coherent integration would be possible in the case in which both the radar and the target were stationary, but this case is not of much practical value. The difference between various types of both detectors and video integrating circuits** is small, as far as results of this kind of study are concerned. There are, of course many reasons why a choice is made in practice, such as sensitivity to small changes in amplifier gains, vulnerability to countermeasures, etc.

It is worth describing one scheme for integrating in which a pulse known to be only noise is subtracted from each possible signal plus noise pulse. N of these composite pulses are then integrated. With no signal, the average value of any number of such composite pulses is nearly zero, so that the required bias level is considerably reduced. Such a method is much less sensitive to a small change in bias level, and would usually be preferred in practice. This case is much more difficult to calculate than the straight addition case; and since sample calculations show the results to be nearly identical, the latter method has been used to obtain the curves of Figures 1 thru 50.

Figures 51 and 52 show the difference in sensitivity to bias level for this method. Figures 53 and 54 show the comparison of straight integration to the case in which a noise pulse is subtracted from each signal-plus-noise pulse.

Practical types of electronic pulse integrators often take the form of very narrow band audio filters having their center frequency at the pulse repetition frequency[(21)] or some harmonic thereof. The action of such a filter can be understood roughly by consideration of the frequency spectrum of a finite group of N pulses. The

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* It is assumed throughout this report that the video bandwidth is large compared with the I F bandwidth. Actually, the results will be affected only if the video bandwidth is small compared with the I F bandwidth[(24)], a condition not often found in practice.

** One might ask if there would be any advantage in having an integrator which adds the sum of the *squares* of the N pulses or perhaps the sum of some other function of the amplitude. Actually, it can be easily shown that this just corresponds to changing the shape of the detector curve, and what is being asked is, "Is any shape of detector curve much superior to the linear or square law form?" Apparently the answer is no. There is a "best" detector curve for every different signal strength, x, given by $\log I_0(v\sqrt{2x})$ where I_0 is a modified Bessel function. No results have been obtained for this detector function, but it is thought that the maximum difference in range between this and the square law or linear detector will not exceed five percent.

envelope of such a spectrum is simply the familiar sin x/x curve of a single pulse, while the actual curve has appreciable values only in the neighborhood of the harmonics of the repetition frequency (including dc)*. The greater N is, the more closely the spectrum clusters around these harmonics. Thus, the filter may be made narrower, excluding more and more noise, but retaining most of the signal energy. With such a narrow band filter-type of integrator it is very simple to subtract a noise pulse from each signal-plus-noise pulse by gating the receiver at a frequency *double* the center frequency of the filter. To prevent the possibility of a signal on every other gate, the sweep length would ordinarily be held at less than one-half of the pulse repetition period. The simple electronic type of integrator has the disadvantage of a fixed integration time. If the number of pulses returned from a target is greater or less than the number of pulses for which the integrator is set, the operation suffers. With the human operator, the story is different. He can adjust his integration time rapidly to fit changing situations. This procedure could be approximated electronically by the use of two or more successive integrators in series,** or by the use of so-called "weighting circuits." Such a complicated procedure does not come within the scope of this report.

METHOD OF OBTAINING THE BIAS LEVEL

By means covered in detail[23], in a separate mathematical report***, the probability that the sum of N pulses of noise voltage alone will be greater than an arbitrary level y is obtained. This relation may be symbolically represented by

$$P_N = f(y) \tag{13}$$

where y is measured in units of the rms value of the noise. The number of groups of noise pulses which are observed in a fixed false alarm time, τ_{fa}, is then found.

When speaking of noise pulses, it is convenient to assume mentally a range gate equal to the pulse length at a fixed range. If the range sweep is continuous, such as with an A scope, the effective number of independent noise pulses observed in one

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* There is a close resemblance between such a spectrum and the diffraction pattern of an N slit grating (see any standard text book on physical optics).

** The advantage of a multistage integrator is that if a signal which is large enough so that the number of pulses which need to be integrated in order to produce a detectable signal occur in a time appreciably less than the total integration time, one of the sub-stages will detect the signal much sooner than will the final stage.

*** It turns out that the functions which describe the probability that the noise alone, or a given strength signal plus noise will have any arbitrary amplitude, are quite complicated and hence only some of the results and general procedures are given in this report. Furthermore, it should be mentioned in passing that the use of the central limit theorem, or the so-called "normal approximation," is not valid until the number of pulses integrated is of the order of 1000. This is because the values of the distribution functions far out on the tails play a major role in the calculations. Several investigators in the past have made the mistake of assuming that the normal approximation was satisfactory if N were only of the order of 10.

repetition period is given by the length of the sweep divided by the pulse length*, hereafter called η. It is apparent that $\eta = 2L/c\tau_p = 10.8L/\tau_p$ where L is the sweep length in miles, c is the velocity of light, and τ_p is the pulse length in microseconds. In the special case in which the sweep occupies the total time between pulses, $\eta = 1/\tau_p f_r$, which is merely the reciprocal of the duty cycle. The time for N pulses to occur is N/f_r. Therefore $\frac{\tau_{fa}}{N/f_r} = \frac{\tau_{fa} f_r}{N}$ groups are observed in the time τ_{fa}, assuming only one gate per sweep. Since the effective number of gates per sweep is η, the total number of independent chances for obtaining a false alarm in τ_{fa} is**

$$n' = \frac{n}{N} = \frac{\tau_{fa} f_r \eta}{N} \tag{14}$$

The false alarm time is defined as the time in which the probability is ½ that the noise will not exceed the bias level.*** From (13) and (14),

$$(1 - P_n)^{n'} = \frac{1}{2} \tag{15}$$

from which y, the bias level, is obtained.

PROBABILITY OF DETECTING A SIGNAL

Having established the value of the bias level, the probability that a signal will exceed this level in a given time, namely the detection time τ_d, must be calculated. The signal is assumed to consist of N integrated pulses. The time of such a pulse group is N/f_r. The number of such groups which occur in τ_d is given by

$$\gamma' = \frac{\gamma}{N} = \frac{\tau_d f_r}{N}. \tag{16}$$

As a corollary to the previous definition of detection, it is now assumed that the signal is detected if any one of the γ' groups of pulses exceeds the bias level. One will ask, at this stage, "Why not count exactly how many times the signal exceeds the

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* If the range gate is much wider than the pulse length, the operation of the integrator will suffer more or less, depending on the exact type of integrator used. This corresponds somewhat to the case of an oscilloscope where the spot does not move by at least its diameter within a pulse length.

** This derivation assumes that the antenna is not scanning. With a scanning antenna, integrating channels must be deposed in angular position as well as in time. In order for (14) to hold, the number of pulses per channel per scan must be equal to or greater than N, the number of pulses which each channel integrates.

*** This very nearly, though not exactly, corresponds to the earlier definition given on p. 71.

bias level?" This would in effect correspond to a two-stage integrator. Such a device is not considered here, though it is easy to make an extension of the present theory to cover this case.

At any range R, the normalized signal strength x is obtained from Eq.(12). The probability that the signal plus noise will exceed any value y for a single group of N integrated pulses is known [23], and may be represented symbolically as

$$P = f(y, x). \tag{17}$$

The probability that at least one of the γ' groups will exceed the bias level y is then

$$P' = 1-(1-P)^{\gamma'}. \tag{18}$$

Notice that γ' must be an integer for the analysis to be strictly correct. It will be satisfactory, however, if one always requires $\gamma' \geqslant 1$*.

EFFECT OF ANTENNA SCANNING

If the antenna is scanning, some modifications of Eq.(16) for γ', the number of groups of pulses integrated, will be necessary [24]. If, with a P P I type of presentation, the antenna moves at an angular velocity ω, and the beam width is B, then the number of pulses per target per scan will be

$$N_{sc} = \frac{Bf_r}{\omega} \tag{19}$$

and (16) is replaced by

$$\gamma' = \frac{\gamma}{N} = \frac{\tau_d f_{sc} N_{sc}}{N} \tag{20}$$

where f_{sc} is the number of scans per second. With a simple type of electronic integrator, N_{sc} must be equal to or greater than N for Eq.(20) to be valid, assuming that the integrator does not hold over from scan to scan. If the integrator does hold over from scan to scan, as an operator partially does, then it is only necessary to have $\gamma' \geqslant 1$ as before. In any case (20) only holds if $\tau_d f_{sc} \geqslant 1$.

If $\tau_d f_{sc} < 1$, then $\gamma' = N_{sc}/N$, which must be equal to or greater than 1.*

* It is always best for γ' to equal 1. In this case the integrator effectively integrates pulses during the whole of the detection time. $\gamma' > 1$ is the case in which the detection time is longer than the integration time. Here the probability for detection is greater than if the detection time were *reduced* to the integration time, but *less* than it would be if the integration time were increased to equal the detection time. The case for $\gamma' < 1$ is that one in which the number of signal pulses occuring are fewer than the number for which the integrator is set. In this case the probability of detection is reduced from the value it would have if the integrator were set for exactly the number of signal pulses which occur. To calculate this latter case would require using N to calculate the bias level as in (15), but the use of some lesser value N' in obtaining (17). This will be done, but results have not been obtained as yet.

PRESENTATION OF THE RESULTS

The results are presented in the form of a set of curves. This is necessary because of the complicated form of the analytical solutions. The parameters involved in the curves are:

P = the probability of detecting a target at range R.
R/R_0 = the ratio of the range to the idealized range.

$n = \tau_{fa} \cdot f_r \cdot \eta$ *
- τ_{fa} = the false alarm time
- f_r = the pulse repetition rate
- η = the number of pulse intervals per sweep.

$\gamma = \tau_d \cdot f_{sc} \cdot N_{sc}$ **
- τ_d = the detection time
- f_{sc} = the scan frequency
- N_{sc} = the number of pulses per scan
- N = the number of pulses integrated.

A summary of the range of the variables for the curves presented will be found on page 84.

AN EXAMPLE WITH A QUASI-STATIONARY TARGET

A simple example is now solved assuming a stationary target. The radar set will also be assumed to be stationary. The following data are taken as given:

ω = angular rate of antenna = 30°/sec, f_{sc} = 1/12
B = beam width of antenna = 3.0°
f_r = pulse repetition rate = 500 per second
τ_p = pulse length = 1 microsecond
R_0 = idealized range for given target and average aspect = 40 miles
τ_{fa} = required false alarm time = 5 minutes
τ_d = required detection time = 25 seconds.

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* $\eta = \tau_{fa}/\tau_p$ if the sweep occupies the total time between pulses.

** See also (16), and the conditions on (20). The notation used on Figs.1-54 is $\gamma = \tau_d \cdot f_r$, which represents the special case in which there is no scanning. In general this should be replaced by $\gamma = \tau_d \cdot f_{sc} \cdot N_{sc}$.

Type of detector - electronic integrator, $N = 50$; sweep length = 20 to 80 miles.

Step 1. Calculate N_{sc} from (19)

$$N_{sc} = \frac{Bf_r}{\omega} = \frac{3\times500}{30} = 50$$

Step 2. Calculate γ from (20)

$$\gamma = \tau_d f_{sc} N_{sc} = 25\times\frac{1}{12}\times50 = 104$$

Step.3. Calculate η from $\eta = 10.8L/\tau_p$ where L is the sweep length in miles and τ_p is the pulse length in microseconds.

$$\eta = \frac{10.8\times(80-20)}{1} = 648$$

Step 4. Calculate n from (14)

$$n = \tau_{fa}\cdot f_r\cdot\eta = (5\times60)\times500\times648 = 0.98\times10^8$$

Step 5. Refer to Fig. 23; $n = 10^8$ and $\gamma = 100$. Mentally interpolate a curve for $N = 50$ between $N = 30$ and $N = 100$. This curve gives probability of detection at any R/R_0. R_0 is given as 40 miles. For instance, $P = 0.50$ at $R/R_0 = 1.07$ or at $R = 43$ miles.

MOVING TARGETS AND/OR RADAR

If there is an appreciable change of range with time between the radar and the target, a limit will ordinarily be set on the number of pulses which can be integrated. This is because the returned pulses will just fail to overlap when the target has moved through a distance $d = \tau_p C/2$ where c is the velocity of light. The effective distance over which the pulses can be assumed to contribute their full amplitude is about ½ this value. If the rate of change of range is v, the time available for integration is

$$\tau_i = \frac{\tau_p C}{4v} . \qquad (21)$$

The maximum number of pulses which can be integrated in this time is

$$N_{max} = \tau_i f_r = \frac{\tau_p f_r c}{4v} . \qquad (22)$$

This quantity N_{max} is the maximum number of pulses that can be integrated,* provided that it is not greater than N_{sc}, the number of pulses per scan. In the case where $N_{max}>N_{sc}$, then N_{sc} is the maximum number of pulses which can be integrated.

In the case of approaching targets, one may be concerned with the probability that a target will be detected by the time it has reached a certain range. Assuming the target to have started its approach at range R_1, the probability that it will have been detected at least once by the time it reaches range R is

$$P = 1-\prod_{R_1}^{R}\left[1-P(R)\right] \tag{23}$$

where R progresses from R_1 to R in units of ΔR. The length of the ΔR intervals and the number of pulses integrated per interval are determined from the considerations given above.

An example follows in which (23) can be reduced to a particularly simple form: Assume a continuously directed beam (no scan) and the target moving toward the radar with a constant range rate v. The finite product in (23) may be approximated by

$$\log_e \prod_{R_1}^{R}\left[1-P(R)\right] \approx \frac{1}{\Delta R}\int_{R_1}^{R}(1-P)\,dR \tag{24}$$

and using $\Delta R = d = \frac{\tau_p c}{4}$ equation (23) becomes

$$P = 1-e^{\frac{4}{\tau_p c}\int_{R_1}^{R}(1-P)\,dR} \tag{25}$$

The integrations necessary in the solution of this type of problem must be performed numerically, using the graphical data of figures 1 to 50.

In problems where the antenna is scanning, equation (23) may be approximated in different ways depending on the exact values of the parameters involved. These are rather simple to work out in any specific case.

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* A system could presumably be built incorporating one or more velocity gates. Such a velocity gate would travel with a given preset velocity. In this case, the relative velocity of the target to that of the gate, $v-v_g$, can be used in Eq. (22) in place of the target velocity, v. The greater the number of velocity gates used, the greater will the probability be that the difference between the target velocity and *some one* of the gates will be very small. Therefore, in this gate the allowable value of N_{max} will be large, and the probability of detection in this gate will be increased.

In any multi-channel receiver, such as this, the number of pulse intervals per sweep, η, must be multiplied by the number of channels in calculating n.

EXACT EFFECT OF THE NUMBER OF PULSES INTEGRATED ON THE RANGE

One might expect that for a given n and a given probability of detection, the range to the fourth power would vary as N, as was stated on page 73. This would be true with coherent integration, but with video integration the variation is between $N^{1/2}$ and N (assuming a threshold signal). This effect is due to the so-called "modulation suppression" of the weak signal by the stronger noise in the process of detection.

Fig. 55 shows the exact variation of the exponent of N, here called θ, as a function of N, and of n, for P fixed at 0.50. The effect of n is seen to be quite small.

Fig. 56 shows the variation of the exponent of N for an incremental change of N as a function of N and n. P is again fixed at 0.50. In both cases, θ approaches 0.5 as N approaches infinity; though much more slowly, in the first case.

APPLICATION OF RESULTS TO CONTINUOUS-WAVE SYSTEM

Though this report is concerned primarily with pulsed systems, the results are directly applicable to continuous-wave systems. To accomplish this, the following new notation is introduced:

P_{av} = the average cw transmitter power.
Δf_{cw} = the combined R F and I F bandwidth of the cw receiver.
η' = the number of separate velocity channels incorporated in the receiver.*

The change-over is then made by means of these substitutions:

Replace E_p by $P_{av}/\Delta f_{cw}$ in R_0

Put $\gamma = \tau_d \cdot \Delta f_{cw}$

Put $n = \tau_{fa} \cdot \Delta f_{cw} \cdot \eta$

N is now to be taken as the number of variates (of duration $1/\Delta f_{cw}$) which are integrated after detection.**

---

* In both the pulse and cw analysis it has been assumed that the range or velocity gates or channels are fixed in position. In the case where such gates sweep as a function of time in order to conserve apparatus (or for any other reason), the analysis is not strictly valid. A good rule-of-thumb is that the gate should move through the amplifier pass band in a time equal to the reciprocal of the amplifier pass band. In this case the effective visibility factor is about 0.8. Curves of the visibility factor for other sweep speeds are given in the Mathematical Appendix[23] (a separate report).

** Integration of N variates *before* detection merely corresponds to narrowing the R F (or I F) bandwidth by a factor of $\frac{1}{N}$.

N must be less than γ for the theory to hold. An optimum cw system is one in which $\gamma = 1$ ($\tau_d = 1/\Delta f_{cw}$), and $N = 1$. This gives the greatest range for a given energy expended during the detection time, τ_d. This corresponds exactly to the case $N = 1$ and $\gamma = 1$ in a pulsed system. If the number of range channels η in the pulsed system is equal to the number of velocity channels η' in the cw system, then the two systems, with $N = 1$ and $\gamma = 1$, will have identical ranges for the same average power.

In either case, if $N > 1$, a larger amount of average power is required, everything else remaining equal. In the pulsed case, reducing N necessitates higher peak powers, which may be impracticable; or it necessitates longer pulse lengths, which reduces possible range-resolution and at the same time aggravates the effect of a fixed Doppler shift due to the narrowing of the receiver pass band. In the cw case, reducing N necessitates a target with reasonably constant velocity so that the signal will not wander in and out of the pass band of the receiver, and also a sufficiently slow scan so that each target "pulse" is at least as long as the reciprocal of the receiver pass band.

RANGE OF VARIABLES FOR FIGURES 1 THRU 50

Fig.	P	R/R_0	n	γ	N
1	variable	variable	10^4	1	1
2	"	"	"	10	1,3,10
3	"	"	"	100	1,3,10,30,100
4	"	"	"	1000	1,3,10,30,100,300,1000
5	"	"	"	1,10,100,1000	1
6	"	"	"	10,100,1000	10
7	"	"	"	100,1000	100
8	"	"	"	N	1,10,100,1000
9	"	"	10^6	1	1
10	"	"	"	10	1,3,10
11	"	"	"	30	1,3,10,30
12	"	"	"	100	1,3,10,30,100
13	"	"	"	300	1,3,10,30,100,300
14	"	"	"	1000	1,3,10,30,100,300,1000
15	"	"	"	1,3,10,30,100,300,1000	1
16	"	"	"	10,30,100,300,1000	10
17	"	"	"	30,100,300,1000	30
18	"	"	"	100,300,1000	100
19	"	"	"	300,1000	300
20	"	"	"	N	1,10,30,100,300,1000
21	"	"	10^8	1	1
22	"	"	"	10	1,3,10
23	"	"	"	100	1,3,10,30,100
24	"	"	"	1000	1,3,10,30,100,300,1000
25	"	"	"	1,10,100,1000	1
26	"	"	"	10,100,1000	10
27	"	"	"	100,1000	100
28	"	"	"	N	1,10,100,1000
29	"	"	10^{10}	1	1
30	"	"	"	10	1,3,10
31	"	"	"	100	1,3,10,30,100
32	"	"	"	1000	1,3,10,30,100,300,1000
33	"	"	"	1,10,100,1000	1
34	"	"	"	10,100,1000	10
35	"	"	"	100,1000	100
36	"	"	"	N	1,10,100,1000
37	"	"	10^{12}	1	1
38	"	"	"	10	1,3,10
39	"	"	"	100	1,3,10,30,100
40	"	"	"	1000	1,3,10,30,100,300,1000
41	"	"	"	1,10,100,1000	1
42	"	"	"	10,100,1000	10
43	"	"	"	100,1000	100
44	"	"	"	N	1,10,100,1000
45	0.50	"	variable	N	1,10,100,1000
46	0.90	"	"	N	1,10,100,1000
47	0.99	"	"	N	1,10,100,1000
48	0.50	"	10^4 10^6 10^8 10^{10}	N - variable	variable
49	0.90	"	10^4 10^6 10^8 10^{10}	N - variable	variable
50	0.99	"	10^4 10^6 10^8 10^{10}	N - variable	variable

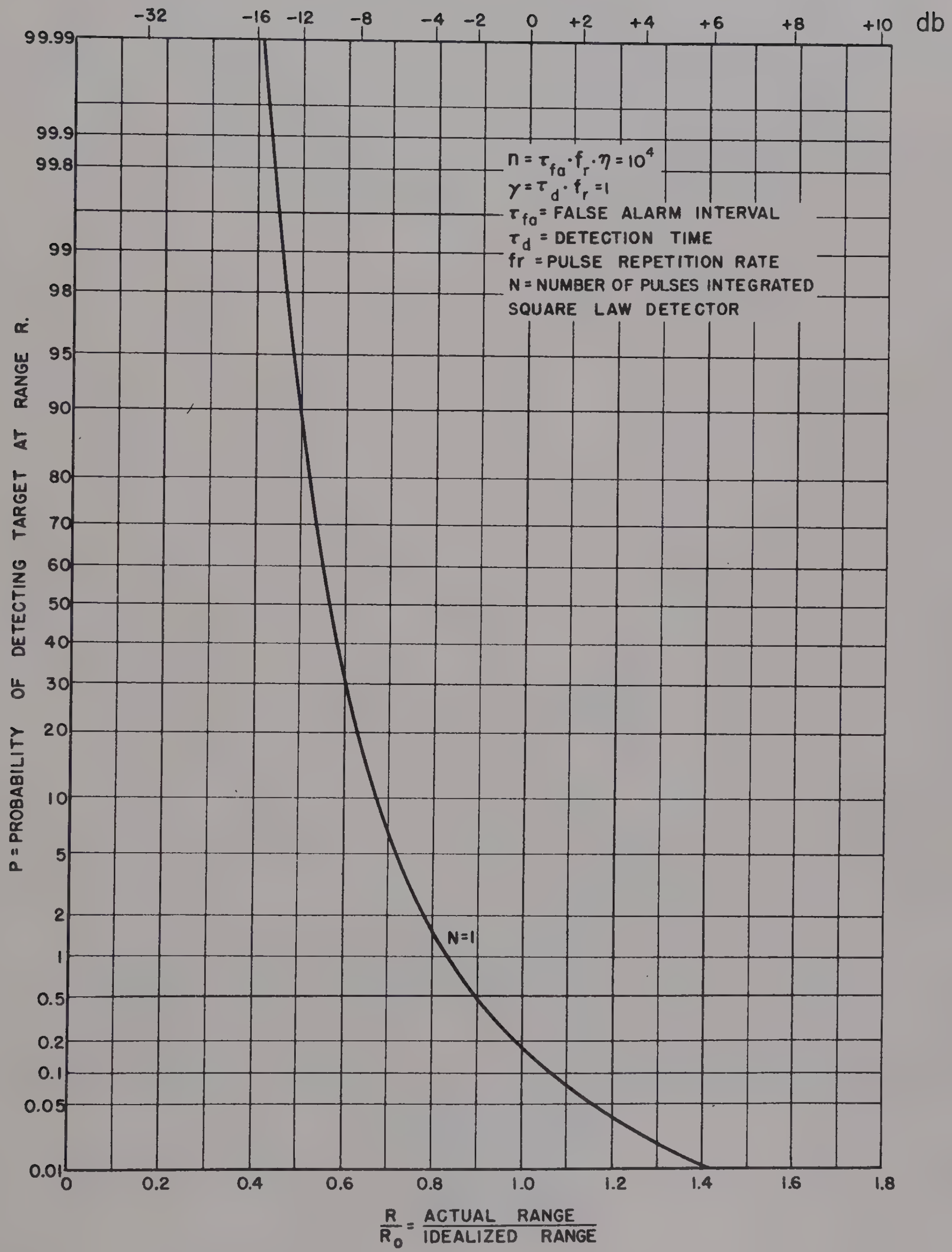

-32 -16 -12 -8 -4 -2 0 +2 +4 +6 +8 +10 db
99.99
99.9
99.8
99
98
95
90
80
70
60
50
40
30
20
10
5
2
1
0.5
0.2
0.1
0.05
0.01
0 0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8
$n = \tau_{fa} \cdot f_r \cdot \eta = 10^4$
$\gamma = \tau_d \cdot f_r = 1$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
N=1
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
$\frac{R}{R_0} = \frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$

FIG. 1

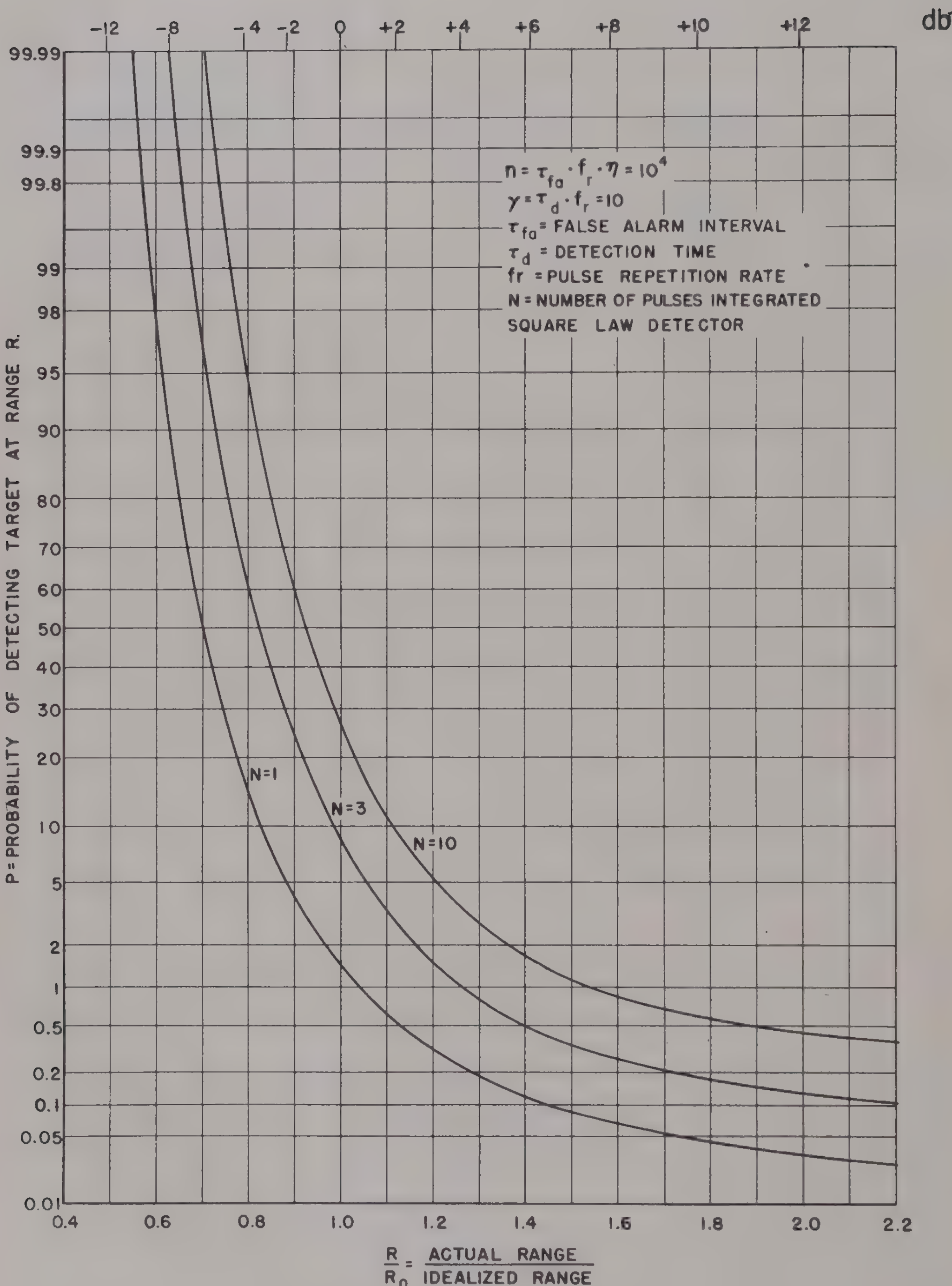
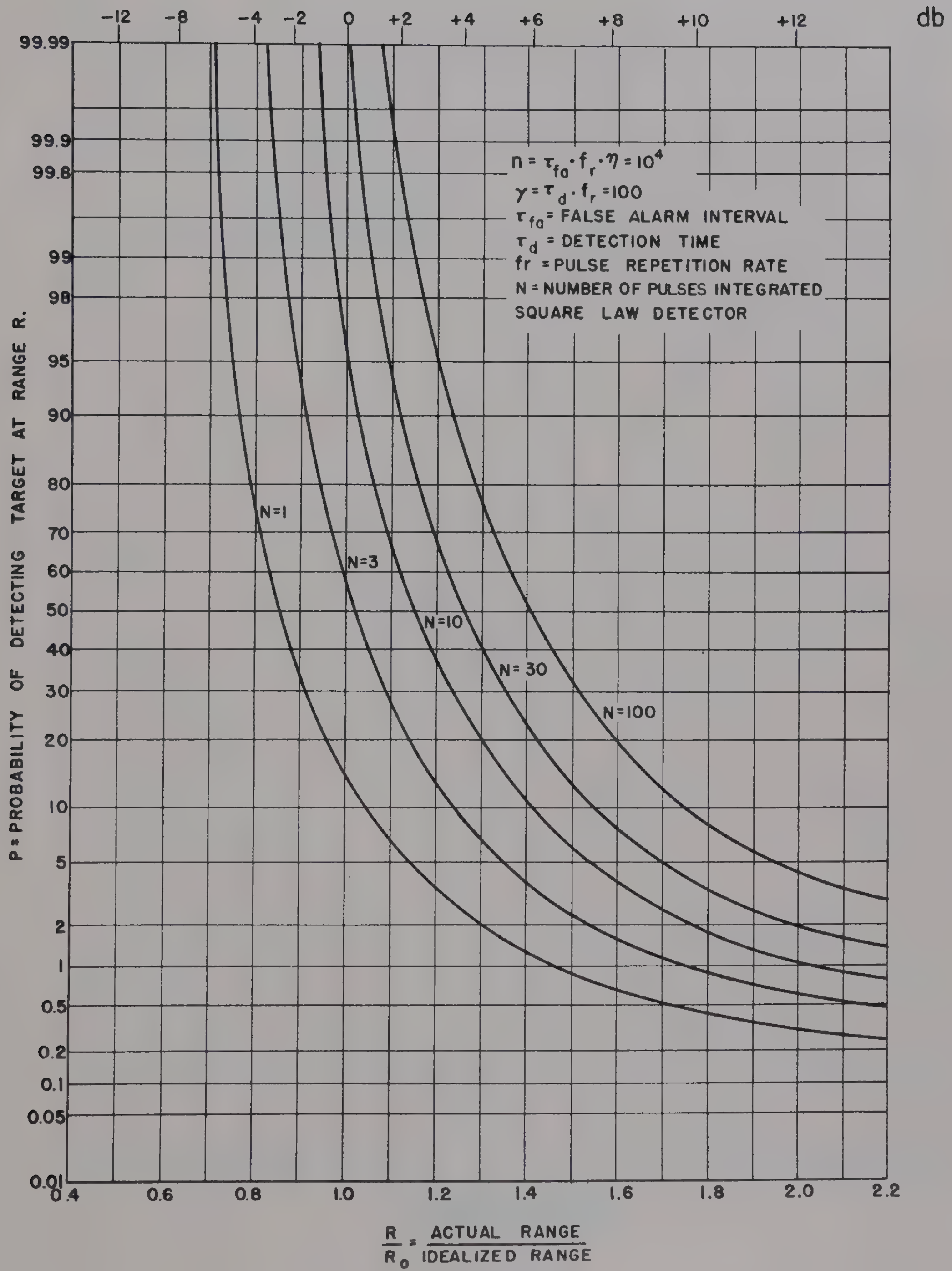
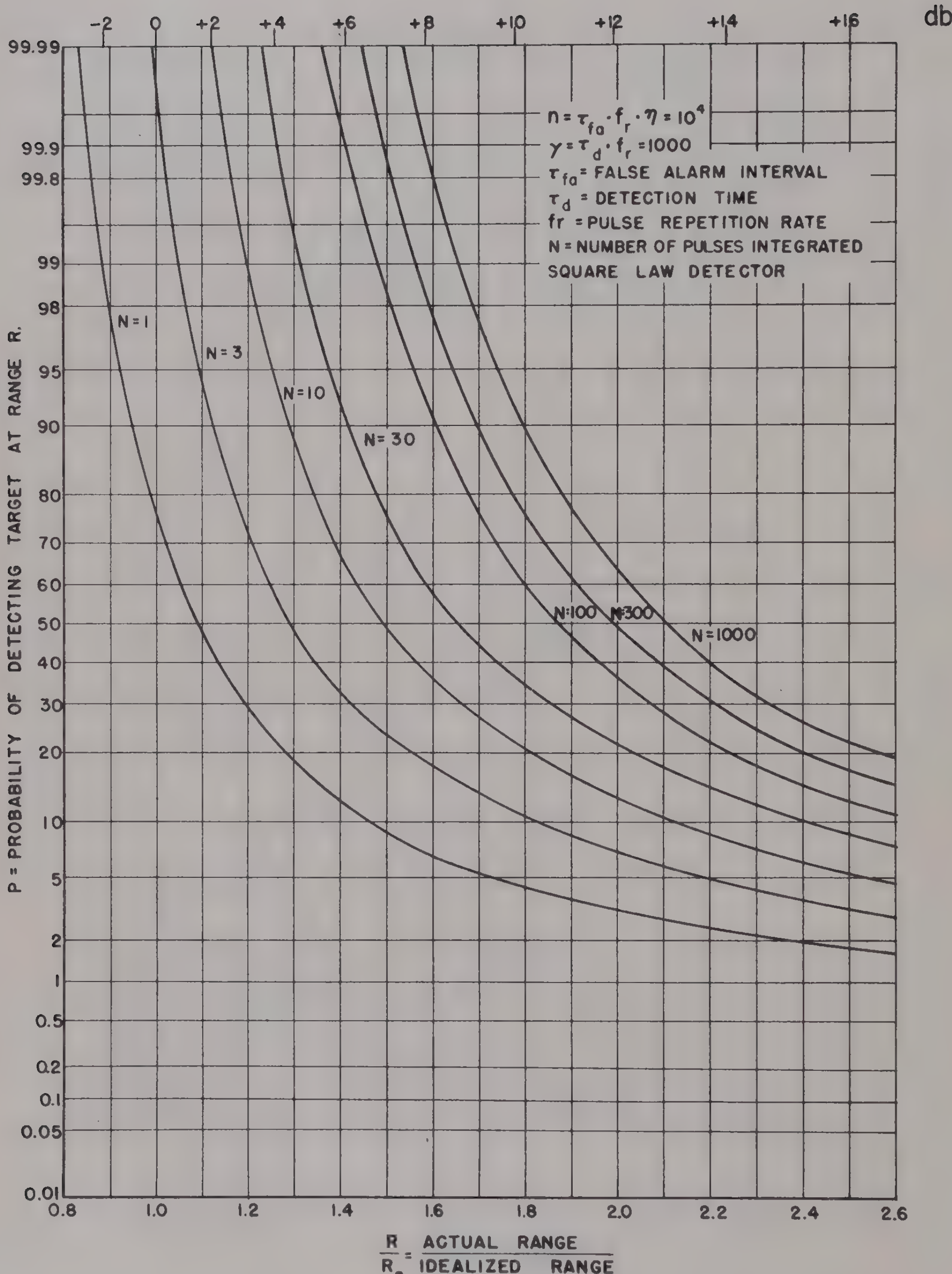

db
-12
-8
-4
-2
0
+2
+4
+6
+8
+10
+12
n = τfa · fr · η = 10⁴
γ = τd · fr = 10
τfa = FALSE ALARM INTERVAL
τd = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
N=1
N=3
N=10
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
99.99
99.9
99.8
99
98
95
90
80
70
60
50
40
30
20
10
5
2
1
0.5
0.2
0.1
0.05
0.01
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
R/R0 = ACTUAL RANGE / IDEALIZED RANGE

FIG. 2



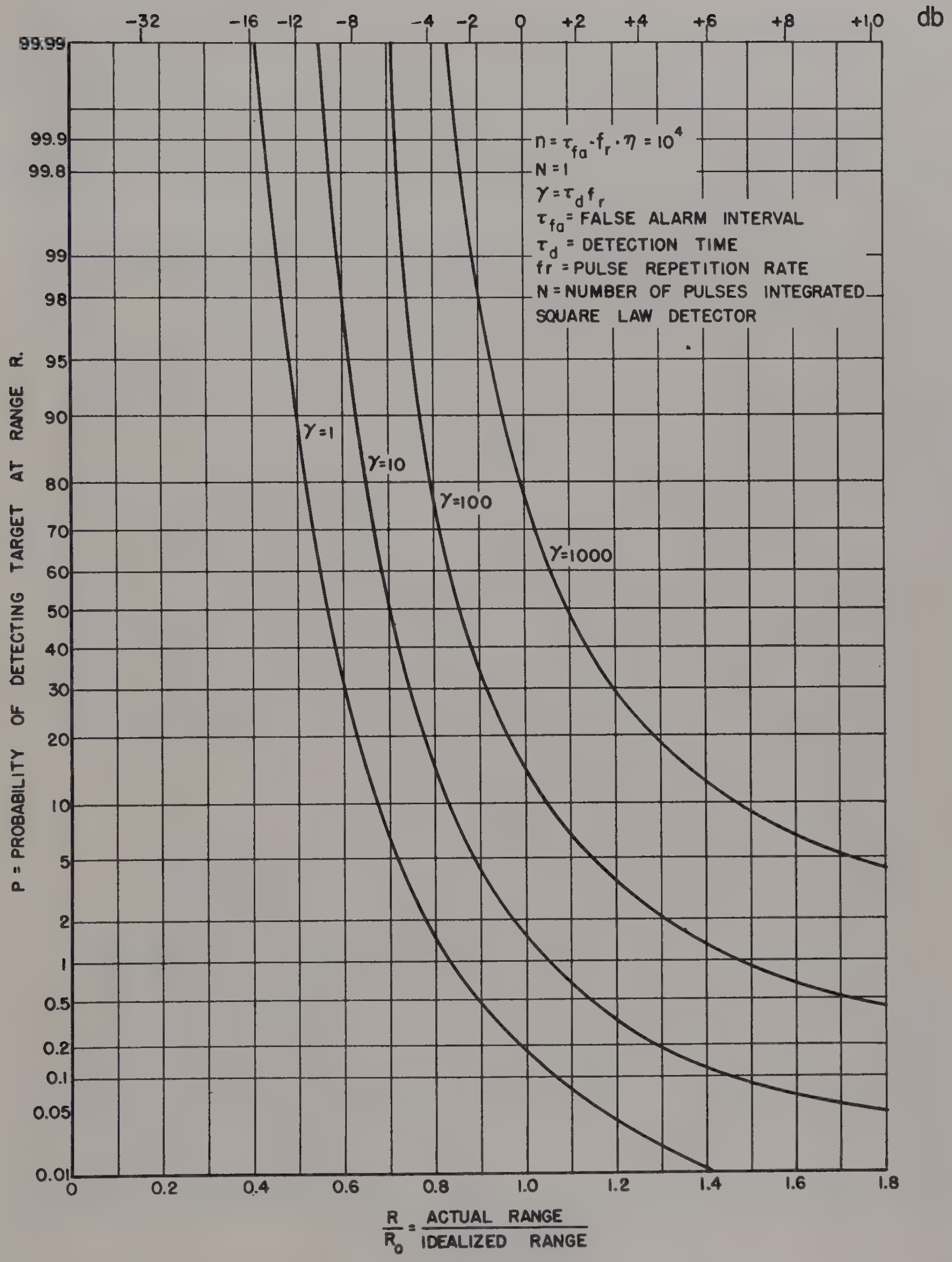
db
-12 -8 -4 -2 0 +2 +4 +6 +8 +10 +12
n = τfa · fr · η = 10^4
γ = τd · fr = 100
τfa = FALSE ALARM INTERVAL
τd = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
N=1
N=3
N=10
N=30
N=100
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
99.99 99.9 99.8 99 98 95 90 80 70 60 50 40 30 20 10 5 2 1 0.5 0.2 0.1 0.05 0.01
0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0 2.2
R/R0 = ACTUAL RANGE / IDEALIZED RANGE

FIG. 3



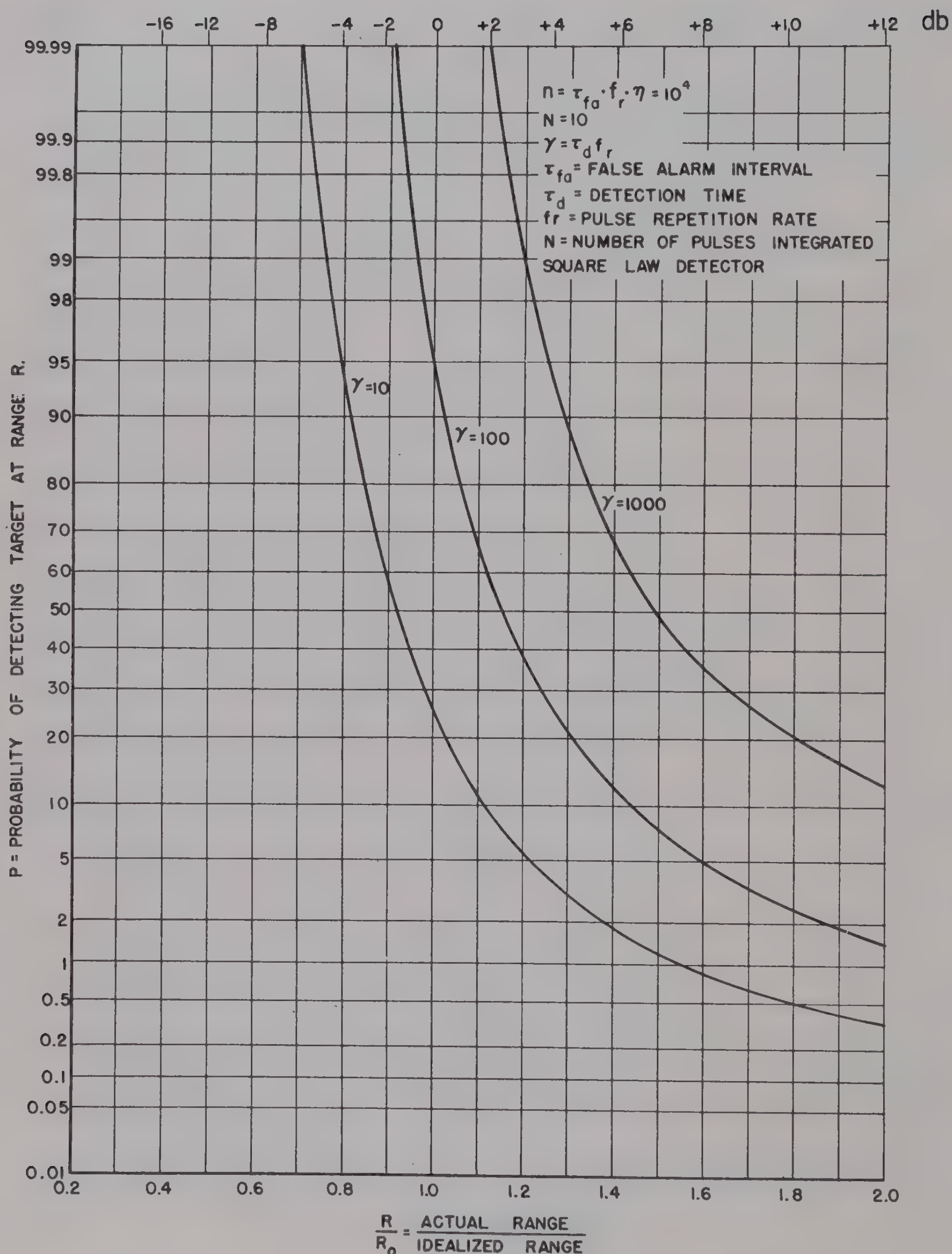
db
-2
0
+2
+4
+6
+8
+10
+12
+14
+16
$n = \tau_{fa} \cdot f_r \cdot \eta = 10^4$
$\gamma = \tau_d \cdot f_r = 1000$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
N = 1
N = 3
N = 10
N = 30
N=100
N=300
N=1000
99.99
99.9
99.8
99
98
95
90
80
70
60
50
40
30
20
10
5
2
1
0.5
0.2
0.1
0.05
0.01
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
0.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
$\frac{R}{R_o} = \frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$

FIG. 4



db
-32
-16
-12
-8
-4
-2
0
+2
+4
+6
+8
+10
$n = \tau_{fa} \cdot f_r \cdot \eta = 10^4$
N = 1
$\gamma = \tau_d f_r$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
γ=1
γ=10
γ=100
γ=1000
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
99.99
99.9
99.8
99
98
95
90
80
70
60
50
40
30
20
10
5
2
1
0.5
0.2
0.1
0.05
0.01
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
$\frac{R}{R_0} = \frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$

FIG. 5



$n = \tau_{fa} \cdot f_r \cdot \eta = 10^4$
N = 10
$\gamma = \tau_d f_r$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
$\gamma = 10$
$\gamma = 100$
$\gamma = 1000$
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
$\frac{R}{R_o} = \frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$
db

FIG. 6

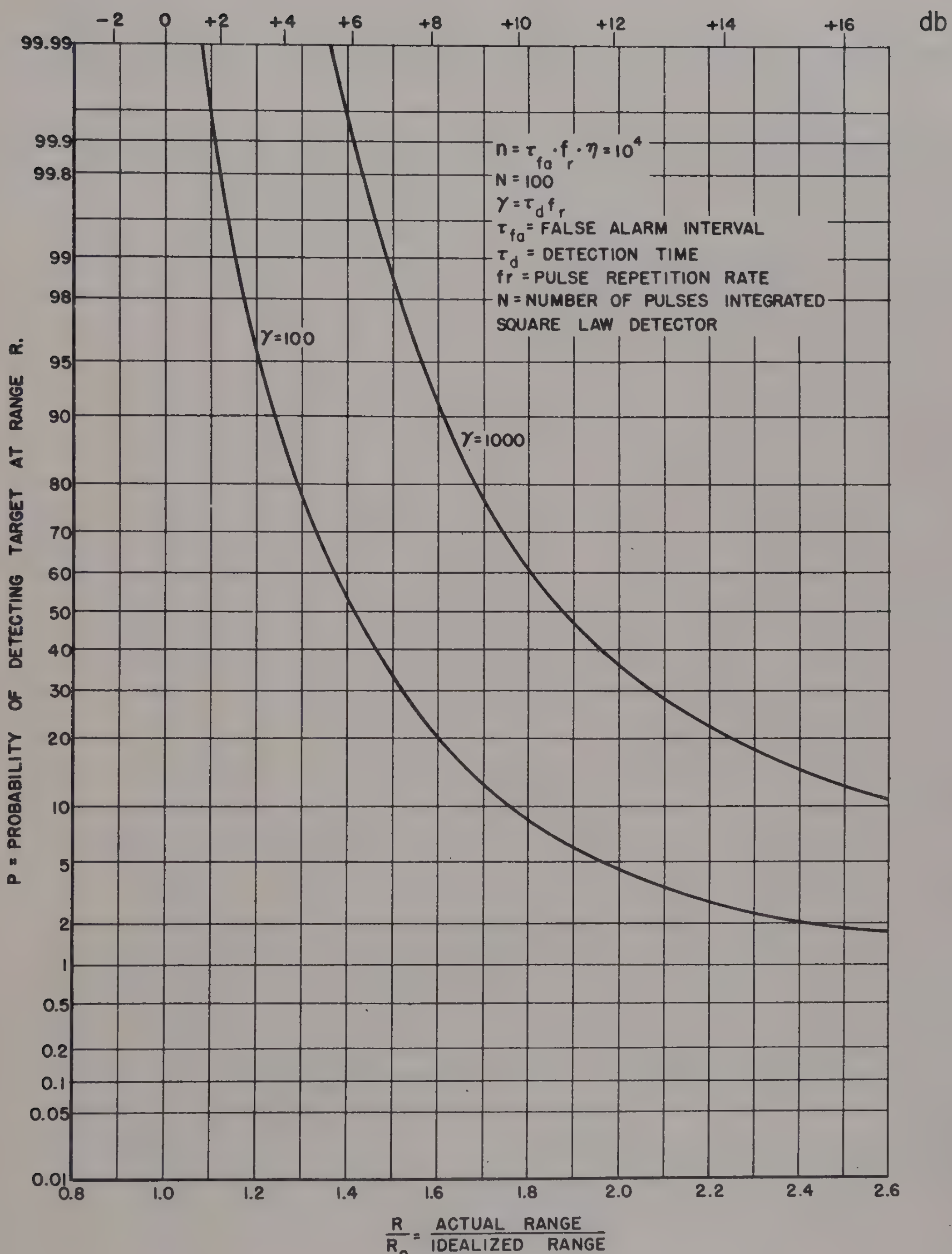
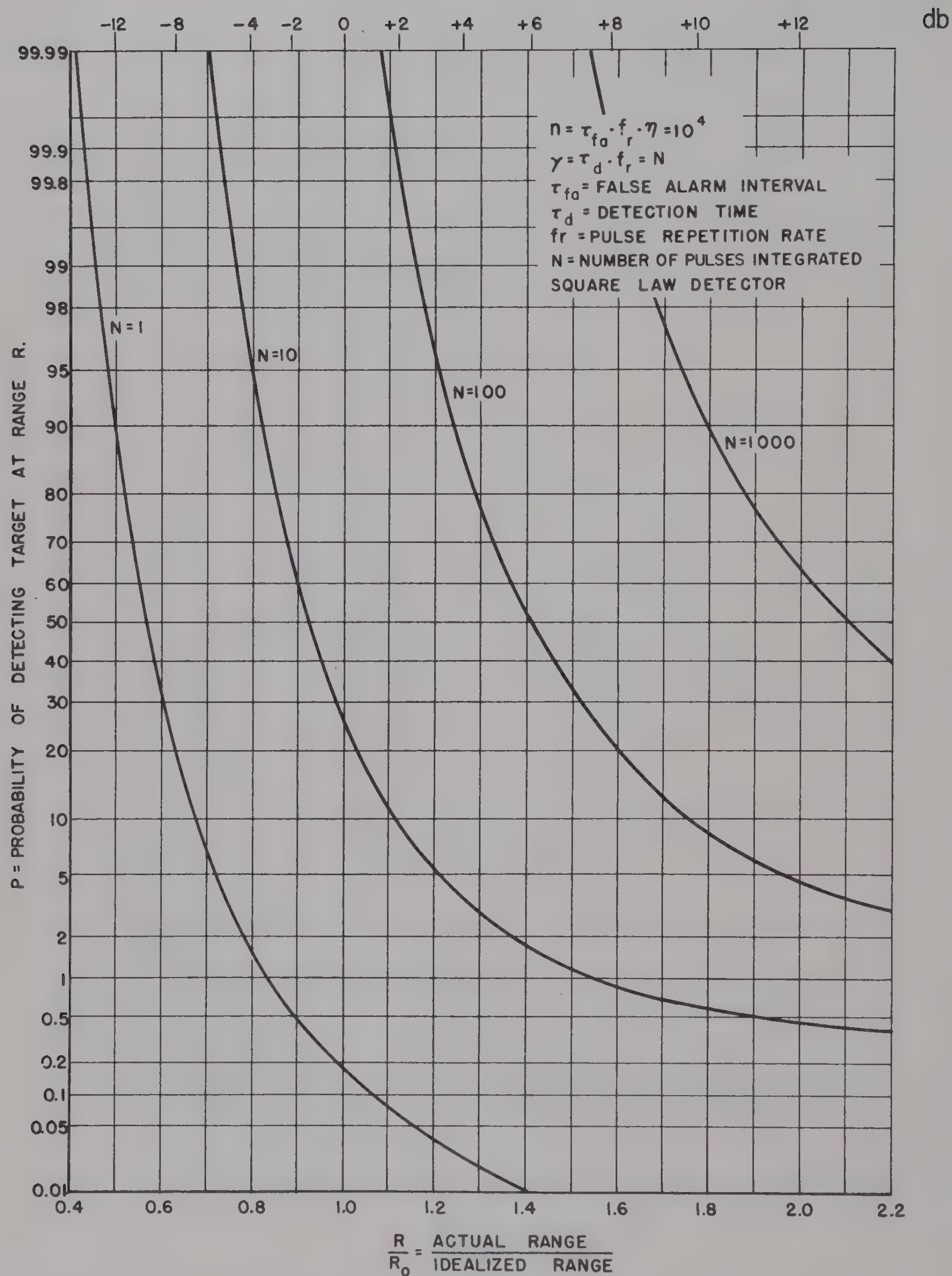
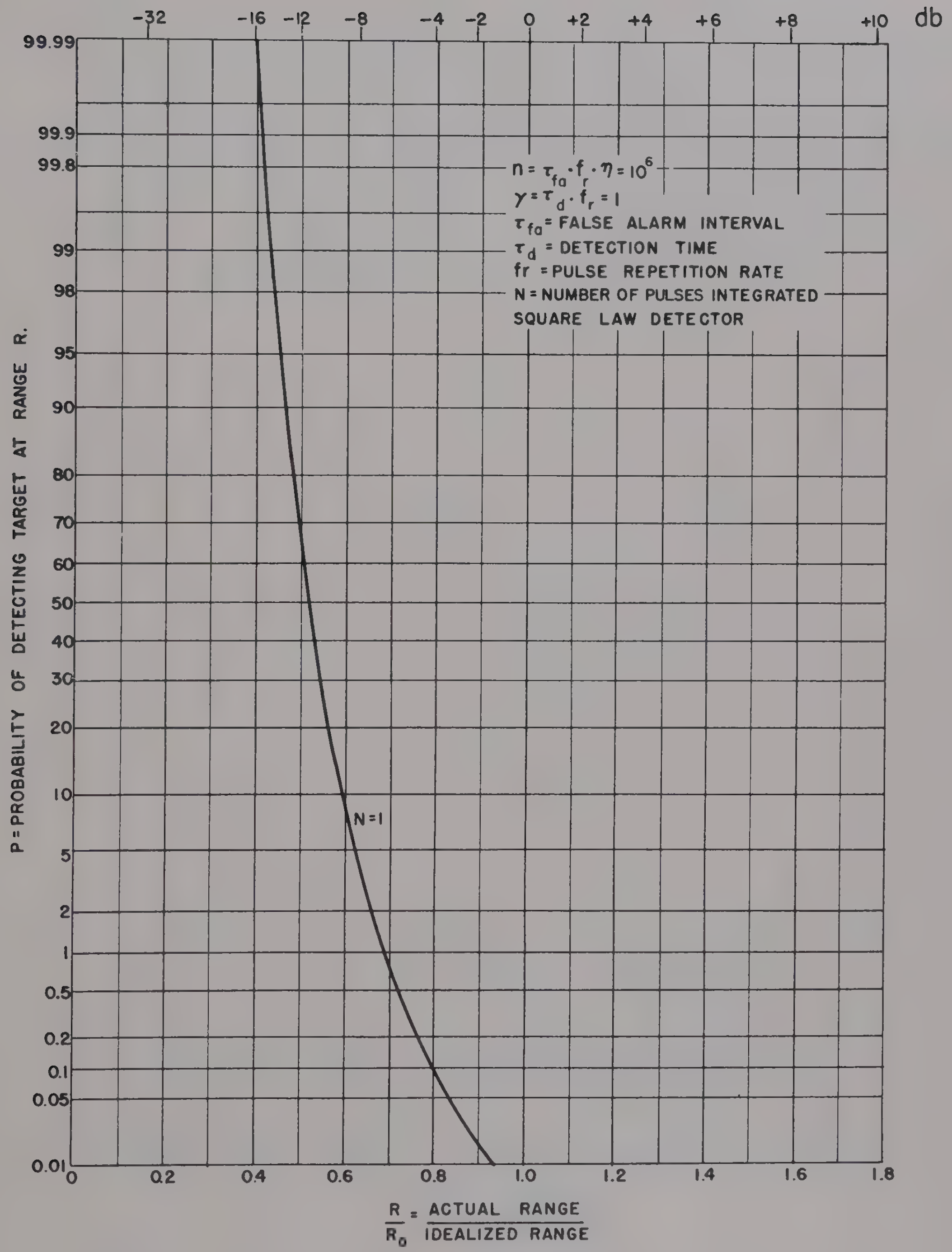

db
−2 0 +2 +4 +6 +8 +10 +12 +14 +16
$n = \tau_{fa} \cdot f_r \cdot \eta = 10^4$
N = 100
$\gamma = \tau_d f_r$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
γ=100
γ=1000
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
99.99 99.9 99.8 99 98 95 90 80 70 60 50 40 30 20 10 5 2 1 0.5 0.2 0.1 0.05 0.01
0.8 1.0 1.2 1.4 1.6 1.8 2.0 2.2 2.4 2.6
$\frac{R}{R_o} = \frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$

FIG. 7



db
-12
-8
-4
-2
0
+2
+4
+6
+8
+10
+12
99.99
99.9
99.8
99
98
95
90
80
70
60
50
40
30
20
10
5
2
1
0.5
0.2
0.1
0.05
0.01
$n = \tau_{fa} \cdot f_r \cdot \eta = 10^4$
$\gamma = \tau_d \cdot f_r = N$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
N = 1
N = 10
N = 100
N = 1000
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
$\frac{R}{R_0} = \frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$

FIG. 8



-32 -16 -12 -8 -4 -2 0 +2 +4 +6 +8 +10 db
99.99
99.9
99.8
99
98
95
90
80
70
60
50
40
30
20
10
5
2
1
0.5
0.2
0.1
0.05
0.01
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
$n = \tau_{fa} \cdot f_r \cdot \eta = 10^6$
$\gamma = \tau_d \cdot f_r = 1$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
N = 1
0 0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8
$\frac{R}{R_0} = \frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$

FIG. 9

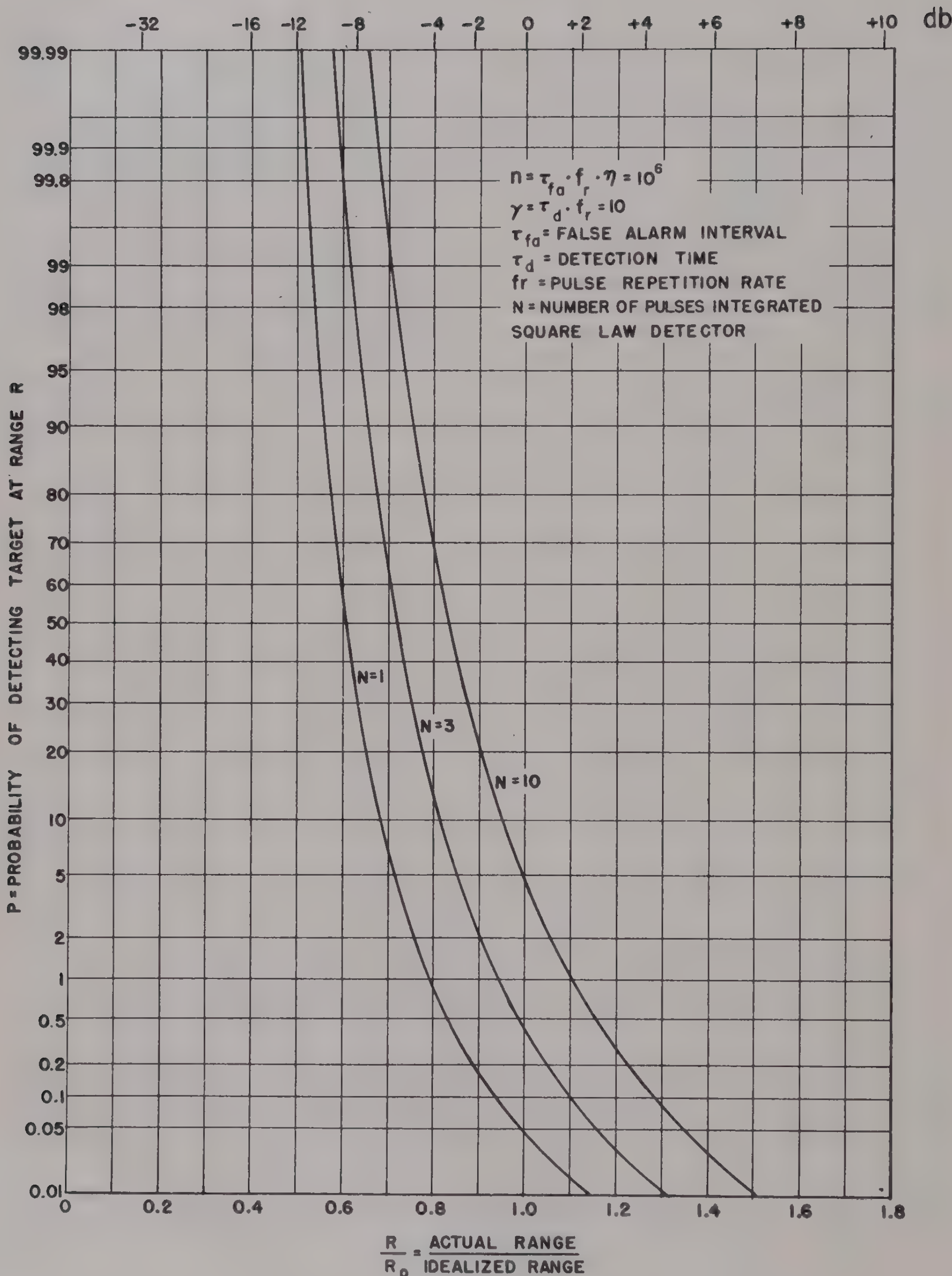
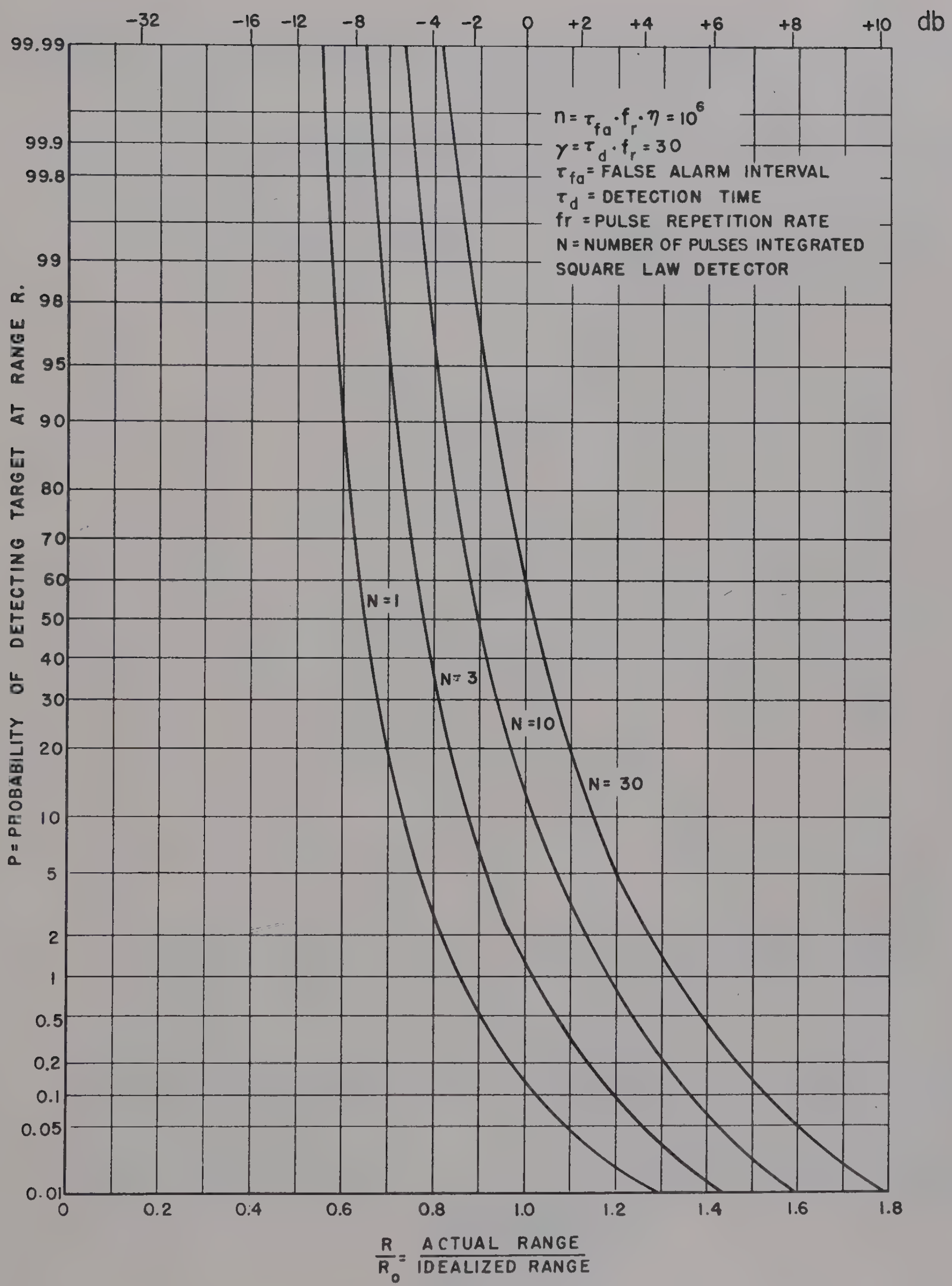
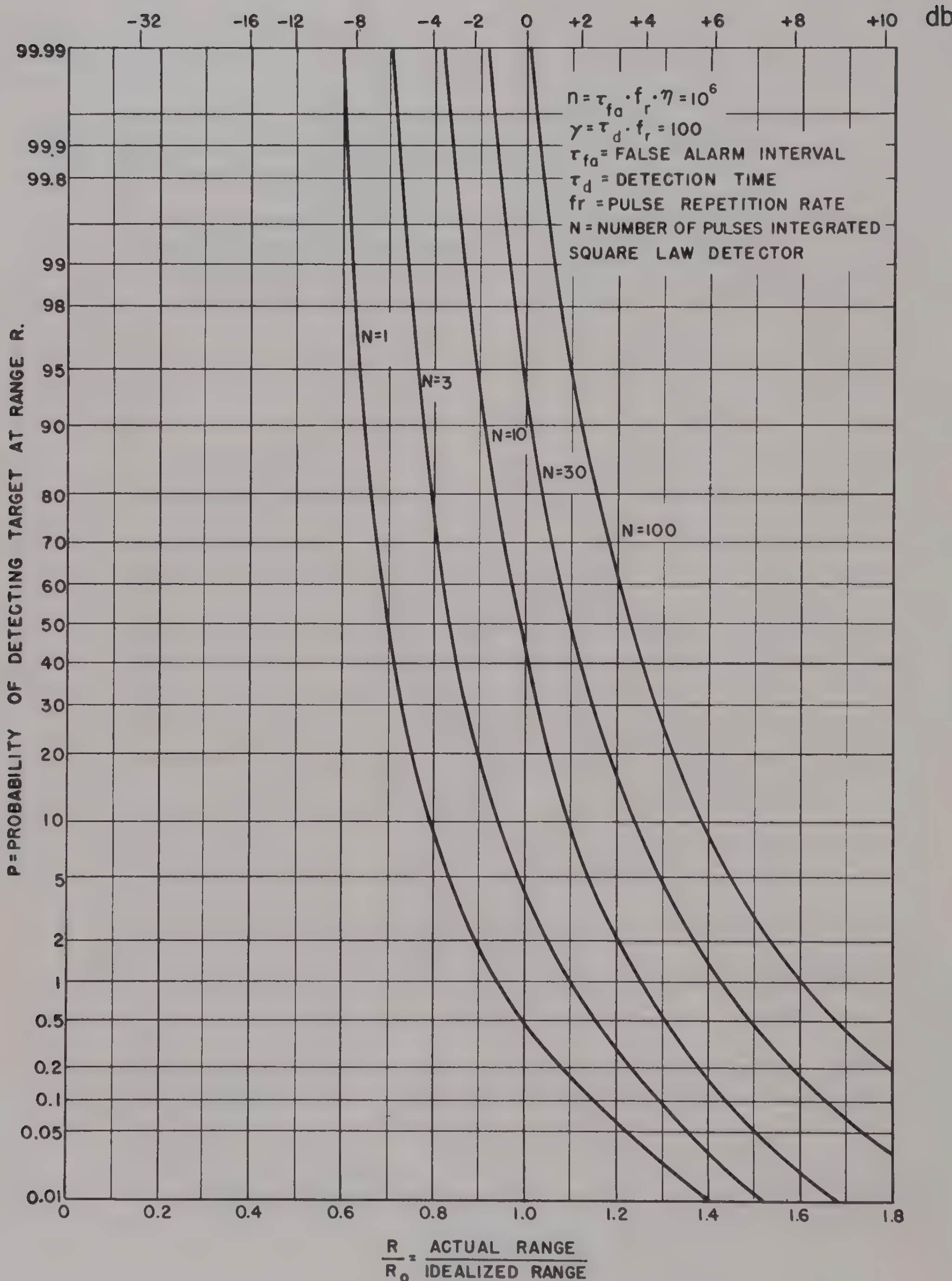

-32 -16 -12 -8 -4 -2 0 +2 +4 +6 +8 +10 db
99.99
99.9
99.8
99
98
95
90
80
70
60
50
40
30
20
10
5
2
1
0.5
0.2
0.1
0.05
0.01
0 0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8
P = PROBABILITY OF DETECTING TARGET AT RANGE R
n = τfa · fr · η = 10^6
γ = τd · fr = 10
τfa = FALSE ALARM INTERVAL
τd = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
N=1
N=3
N=10
R/R0 = ACTUAL RANGE / IDEALIZED RANGE

FIG. 10



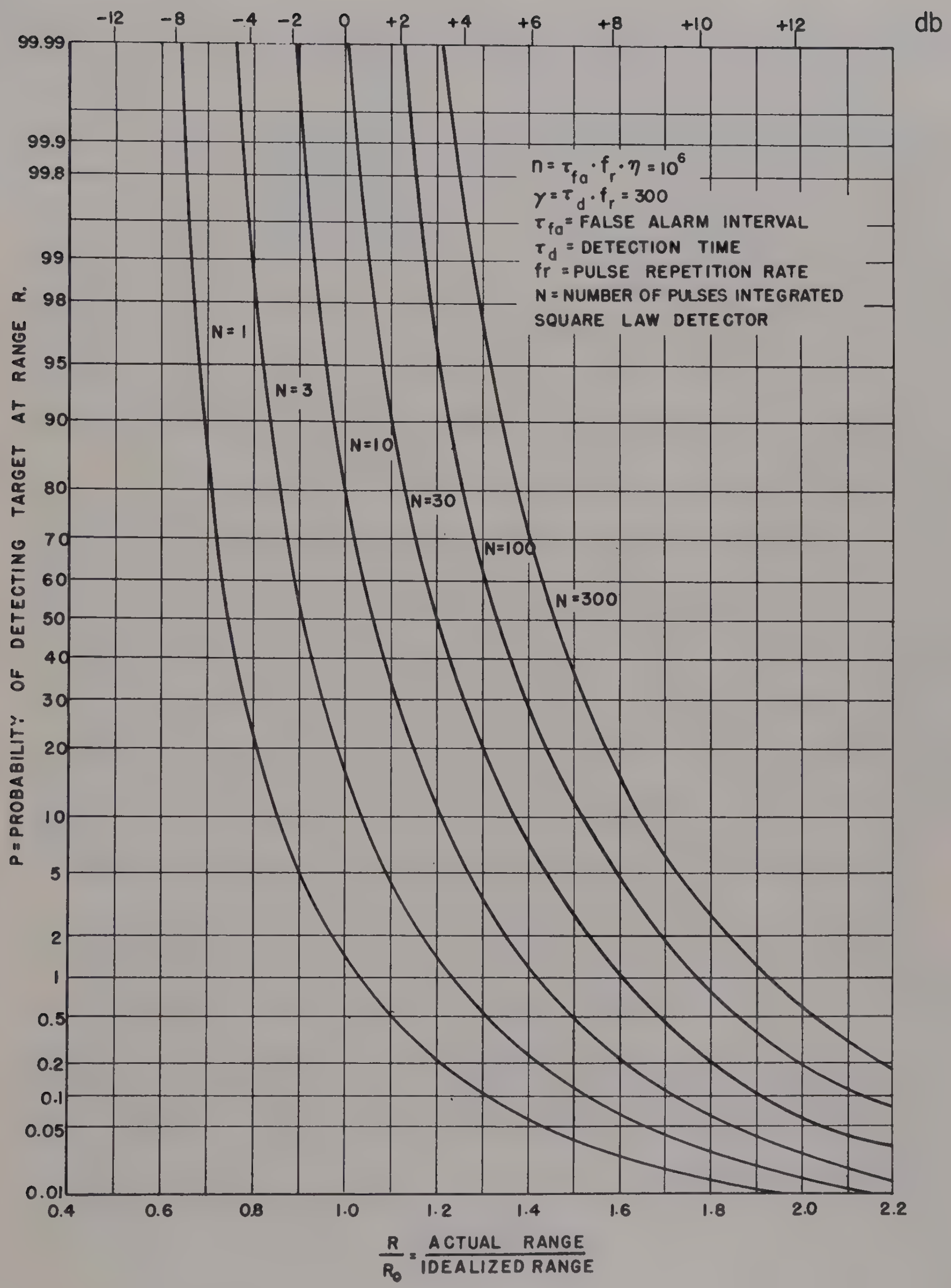
-32 -16 -12 -8 -4 -2 0 +2 +4 +6 +8 +10 db
$n = \tau_{fa} \cdot f_r \cdot \eta = 10^6$
$\gamma = \tau_d \cdot f_r = 30$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
N = 1
N = 3
N = 10
N = 30
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
99.99 99.9 99.8 99 98 95 90 80 70 60 50 40 30 20 10 5 2 1 0.5 0.2 0.1 0.05 0.01
0 0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8
$\frac{R}{R_o} = \frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$

FIG. 11



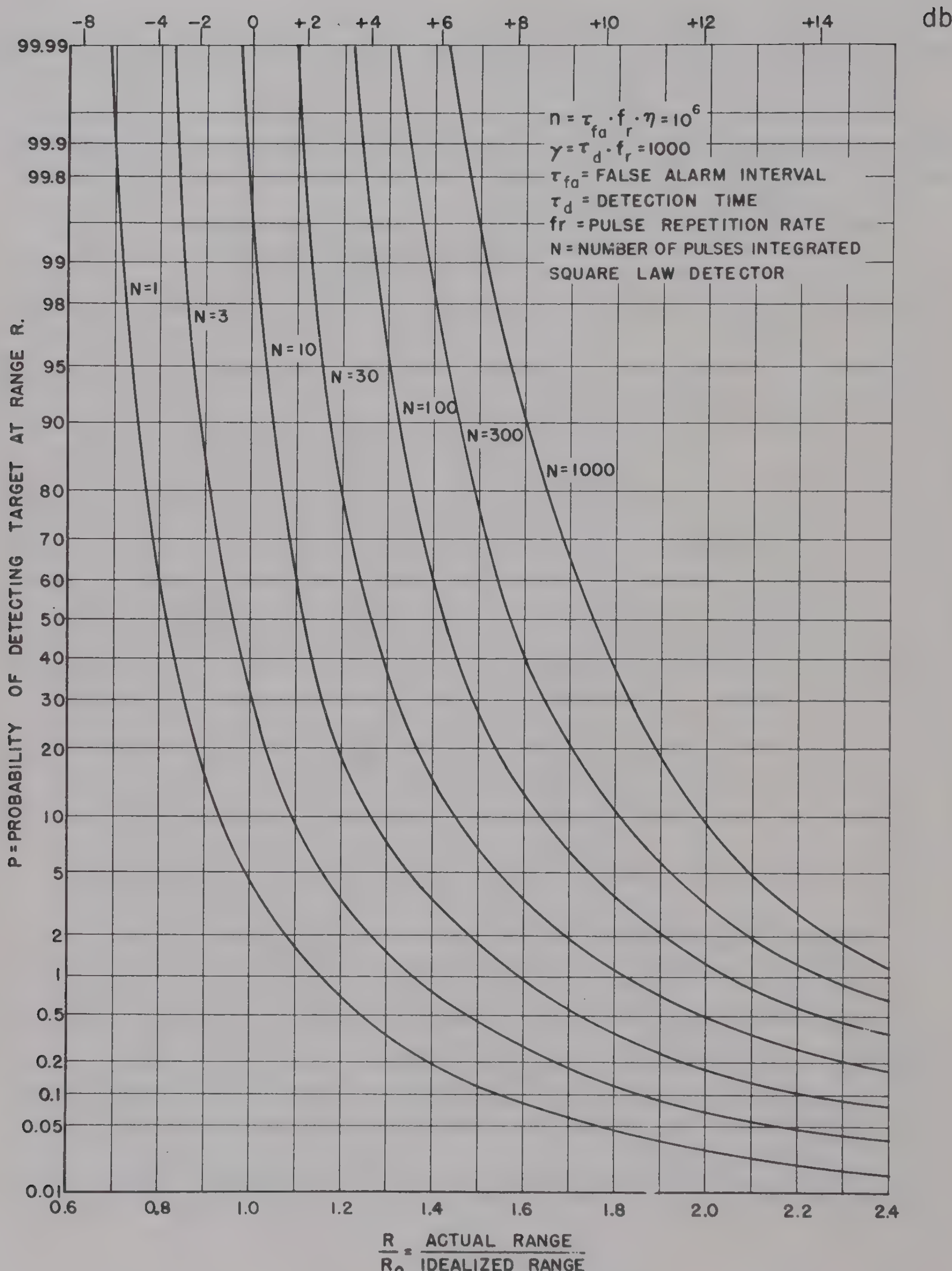
db
-32
-16
-12
-8
-4
-2
0
+2
+4
+6
+8
+10
$n = \tau_{fa} \cdot f_r \cdot \eta = 10^6$
$\gamma = \tau_d \cdot f_r = 100$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
N=1
N=3
N=10
N=30
N=100
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
99.99
99.9
99.8
99
98
95
90
80
70
60
50
40
30
20
10
5
2
1
0.5
0.2
0.1
0.05
0.01
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
$\frac{R}{R_0} = \frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$

FIG. 12



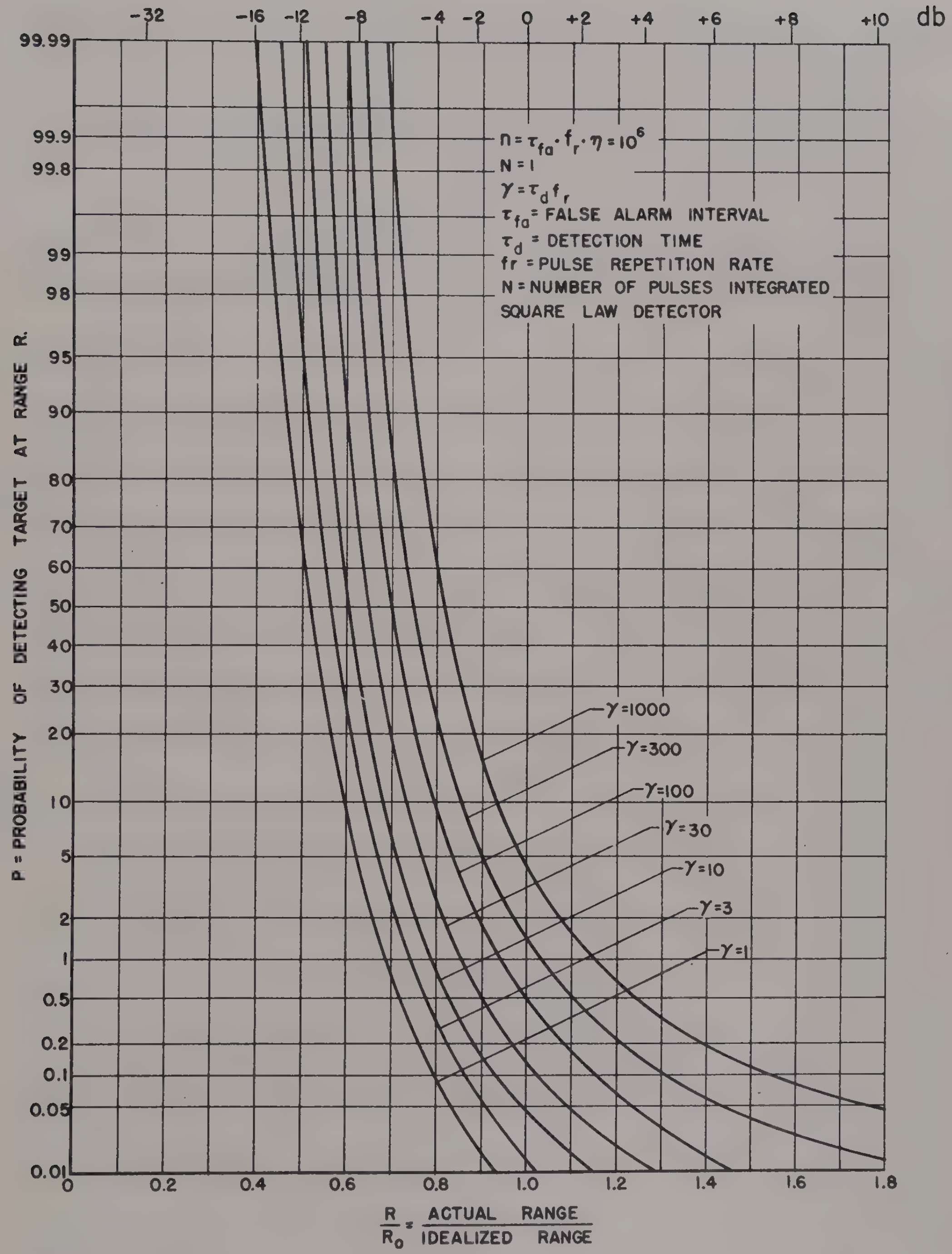
db
-12
-8
-4
-2
0
+2
+4
+6
+8
+10
+12
99.99
99.9
99.8
99
98
95
90
80
70
60
50
40
30
20
10
5
2
1
0.5
0.2
0.1
0.05
0.01
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
$n = \tau_{fa} \cdot f_r \cdot \eta = 10^6$
$\gamma = \tau_d \cdot f_r = 300$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
N = 1
N = 3
N=10
N=30
N=100
N=300
$\frac{R}{R_0} = \frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$

FIG. 13



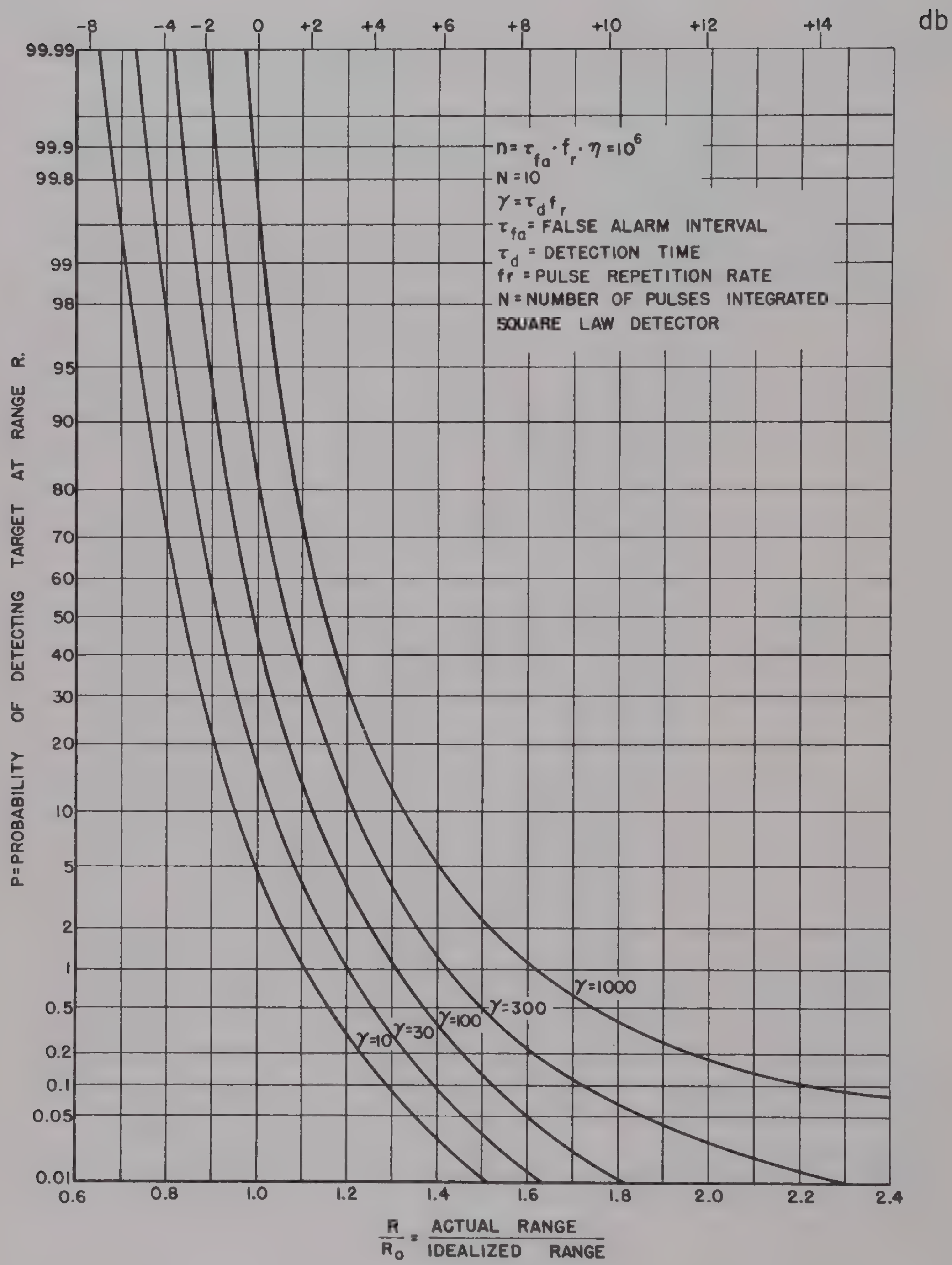
db
-8
-4
-2
0
+2
+4
+6
+8
+10
+12
+14
$n = \tau_{fa} \cdot f_r \cdot \eta = 10^6$
$\gamma = \tau_d \cdot f_r = 1000$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
N=1
N=3
N=10
N=30
N=100
N=300
N=1000
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
99.99
99.9
99.8
99
98
95
90
80
70
60
50
40
30
20
10
5
2
1
0.5
0.2
0.1
0.05
0.01
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
$\frac{R}{R_0} = \frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$

FIG. 14



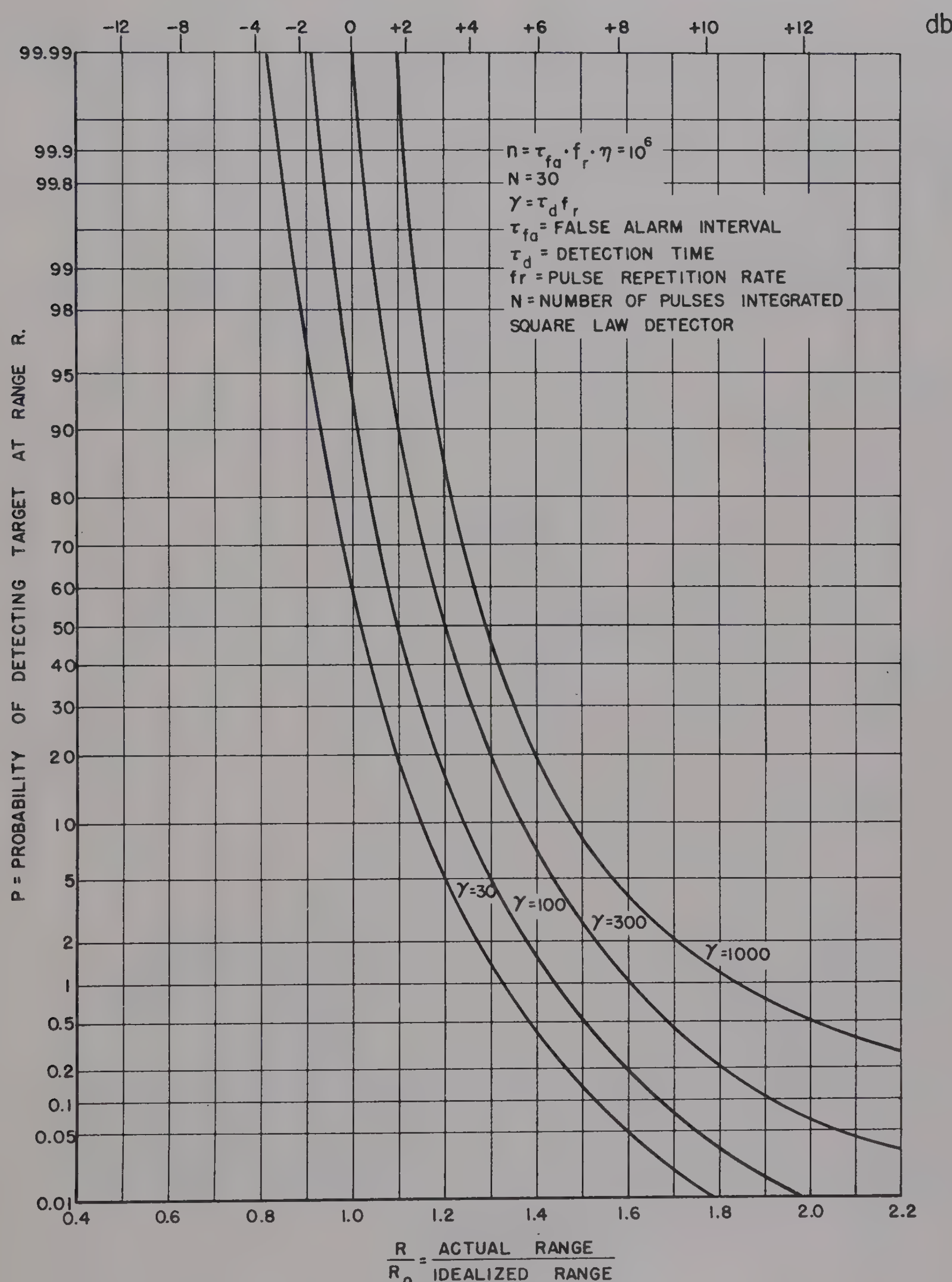
-32 -16 -12 -8 -4 -2 0 +2 +4 +6 +8 +10 db
99.99
99.9
99.8
99
98
95
90
80
70
60
50
40
30
20
10
5
2
1
0.5
0.2
0.1
0.05
0.01
0 0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8
$n = \tau_{fa} \cdot f_r \cdot \eta = 10^6$
N = 1
$\gamma = \tau_d f_r$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
γ=1000
γ=300
γ=100
γ=30
γ=10
γ=3
γ=1
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
$\frac{R}{R_0}$ = $\frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$

FIG. 15



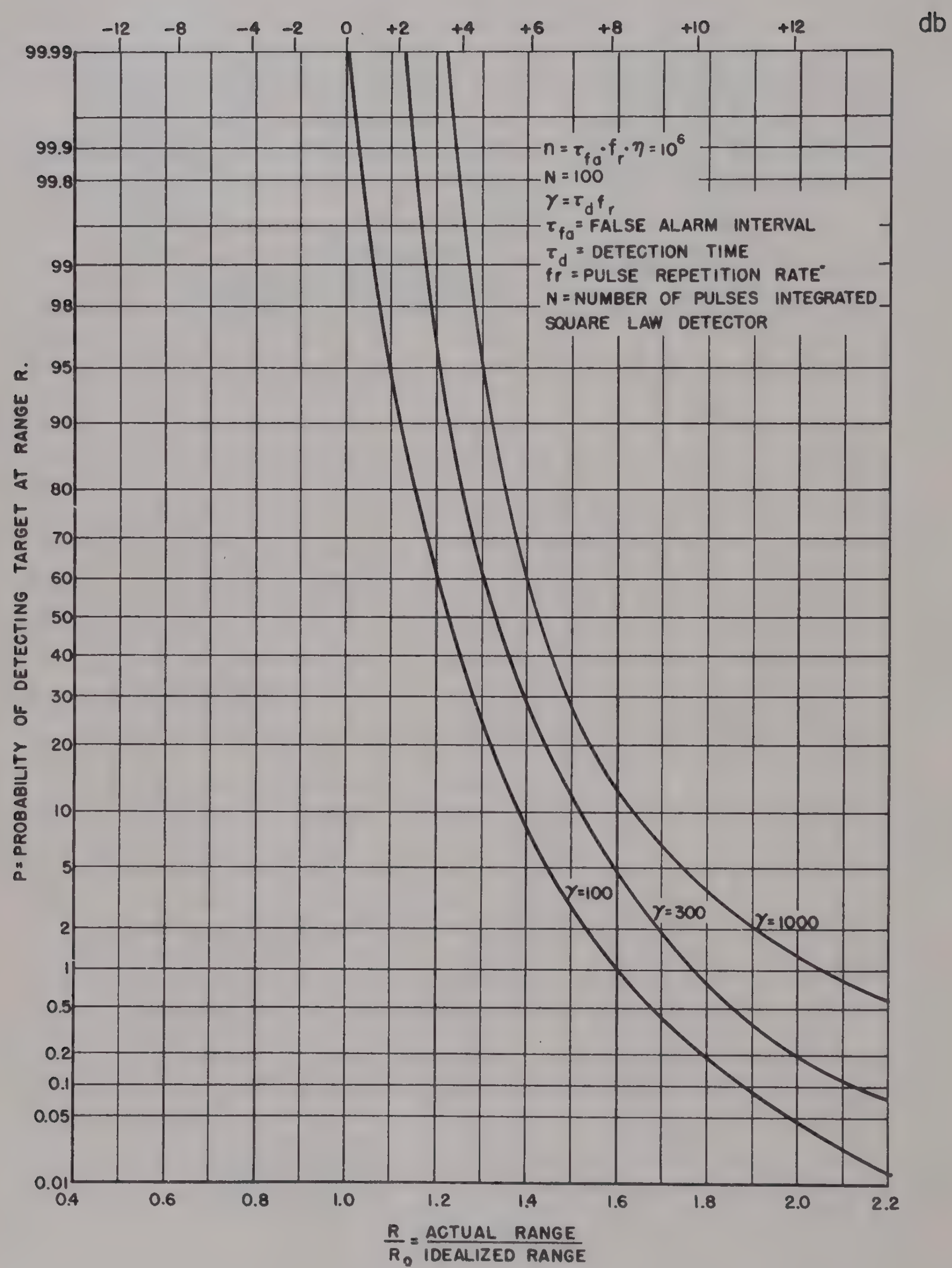
db
-8
-4
-2
0
+2
+4
+6
+8
+10
+12
+14
99.99
99.9
99.8
99
98
95
90
80
70
60
50
40
30
20
10
5
2
1
0.5
0.2
0.1
0.05
0.01
P=PROBABILITY OF DETECTING TARGET AT RANGE R.
$n = \tau_{fa} \cdot f_r \cdot \eta = 10^6$
N = 10
$\gamma = \tau_d f_r$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
γ=1000
γ=300
γ=100
γ=30
γ=10
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
$\frac{R}{R_0} = \frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$

FIG. 16



db
-12
-8
-4
-2
0
+2
+4
+6
+8
+10
+12
99.99
99.9
99.8
99
98
95
90
80
70
60
50
40
30
20
10
5
2
1
0.5
0.2
0.1
0.05
0.01
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
$n = \tau_{fa} \cdot f_r \cdot \eta = 10^6$
N = 30
$\gamma = \tau_d f_r$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
γ=30
γ=100
γ=300
γ=1000
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
$\frac{R}{R_0} = \frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$

FIG. 17



db
-12 -8 -4 -2 0 +2 +4 +6 +8 +10 +12
99.99 99.9 99.8 99 98 95 90 80 70 60 50 40 30 20 10 5 2 1 0.5 0.2 0.1 0.05 0.01
0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0 2.2
n = τfa·fr·η = 10^6
N = 100
γ = τd fr
τfa = FALSE ALARM INTERVAL
τd = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
γ=100
γ=300
γ=1000
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
R/R0 = ACTUAL RANGE / IDEALIZED RANGE

FIG. 18

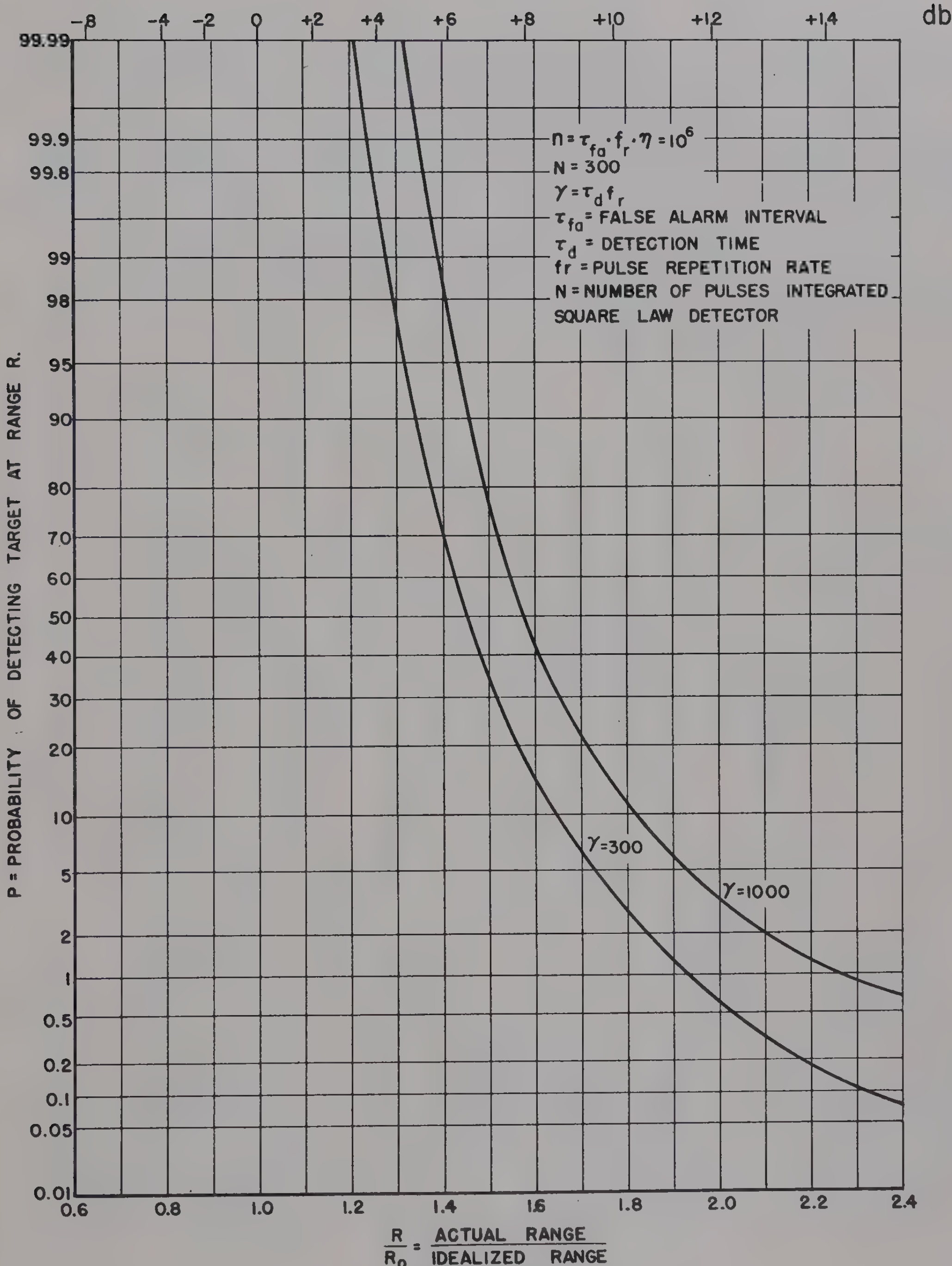
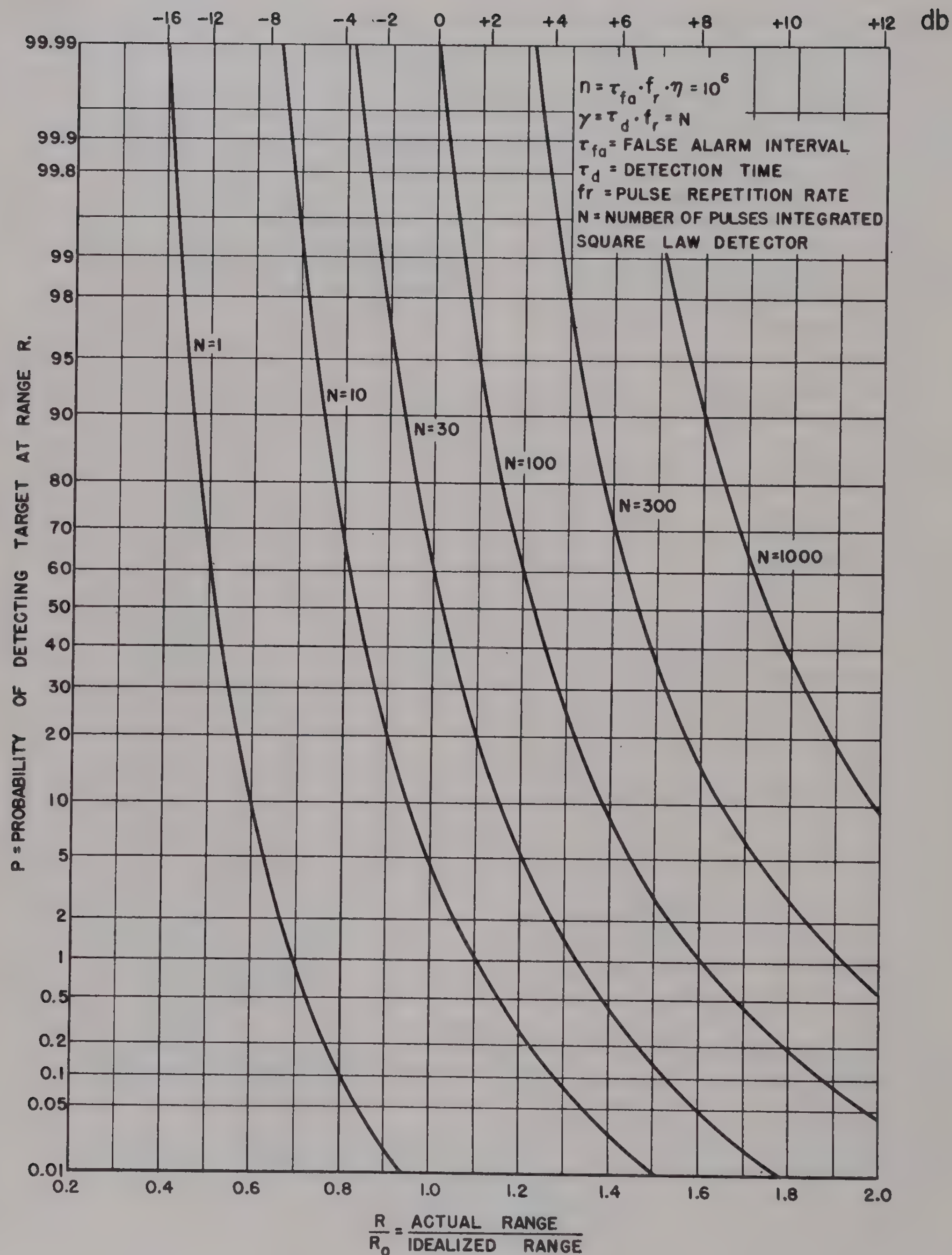

db
-8
-4
-2
0
+2
+4
+6
+8
+10
+12
+14
n = τfa · fr · η = 10^6
N = 300
γ = τd fr
τfa = FALSE ALARM INTERVAL
τd = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
γ=300
γ=1000
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
99.99
99.9
99.8
99
98
95
90
80
70
60
50
40
30
20
10
5
2
1
0.5
0.2
0.1
0.05
0.01
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
R/Ro = ACTUAL RANGE / IDEALIZED RANGE

FIG. 19



-16 -12 -8 -4 -2 0 +2 +4 +6 +8 +10 +12 db
$n = \tau_{fa} \cdot f_r \cdot \eta = 10^6$
$\gamma = \tau_d \cdot f_r = N$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
N=1
N=10
N=30
N=100
N=300
N=1000
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
99.99 99.9 99.8 99 98 95 90 80 70 60 50 40 30 20 10 5 2 1 0.5 0.2 0.1 0.05 0.01
0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0
$\frac{R}{R_0} = \frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$

FIG. 20

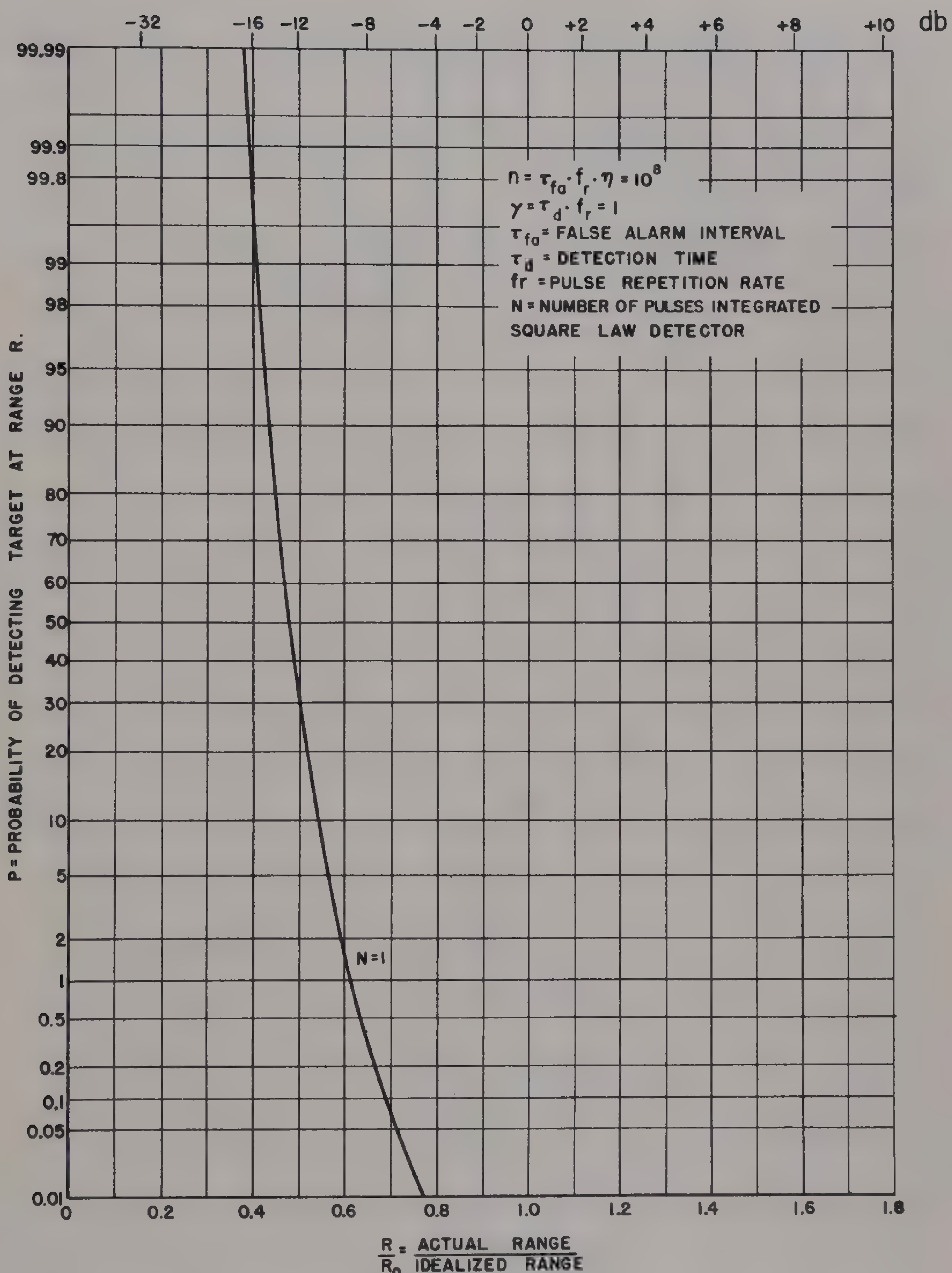
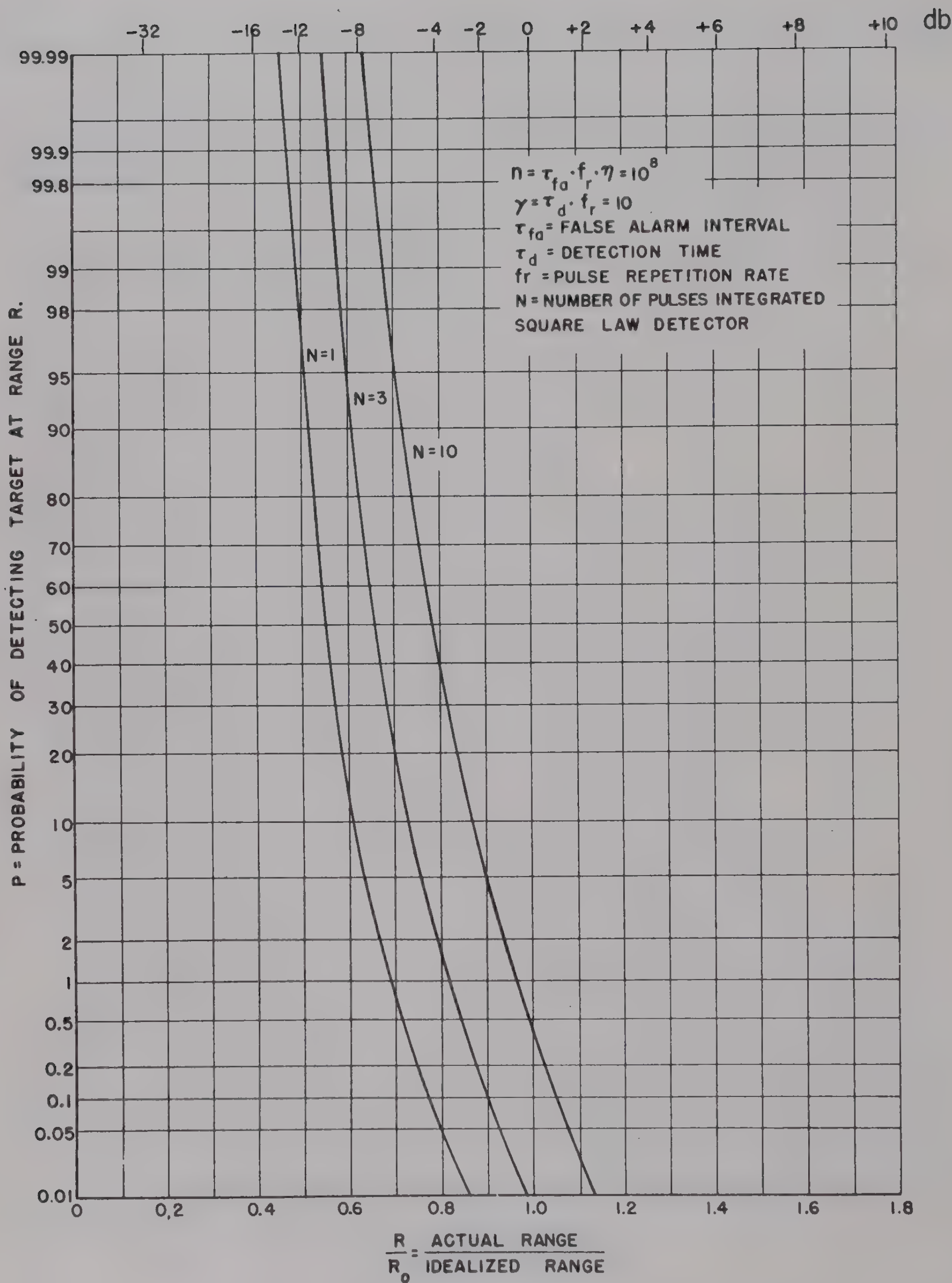

n = τ_fa · f_r · η = 10^8
γ = τ_d · f_r = 1
τ_fa = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
N=1
db
-32 -16 -12 -8 -4 -2 0 +2 +4 +6 +8 +10
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
99.99 99.9 99.8 99 98 95 90 80 70 60 50 40 30 20 10 5 2 1 0.5 0.2 0.1 0.05 0.01
0 0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8
R/R0 = ACTUAL RANGE / IDEALIZED RANGE

FIG. 21



-32
-16
-12
-8
-4
-2
0
+2
+4
+6
+8
+10
db
99.99
99.9
99.8
99
98
95
90
80
70
60
50
40
30
20
10
5
2
1
0.5
0.2
0.1
0.05
0.01
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
$n = \tau_{fa} \cdot f_r \cdot \eta = 10^8$
$\gamma = \tau_d \cdot f_r = 10$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
N=1
N=3
N=10
0
0,2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
$\frac{R}{R_o} = \frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$

FIG. 22

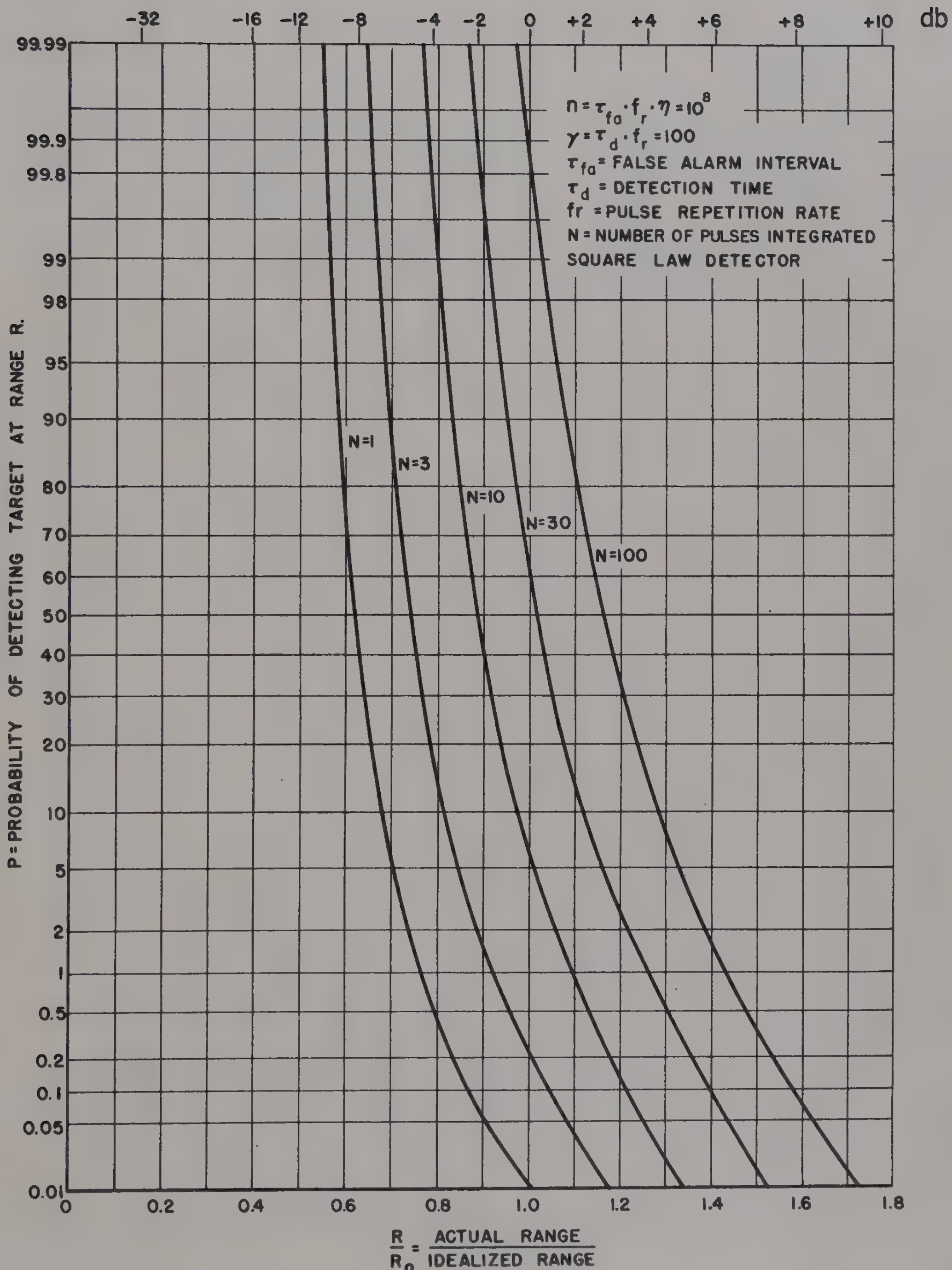
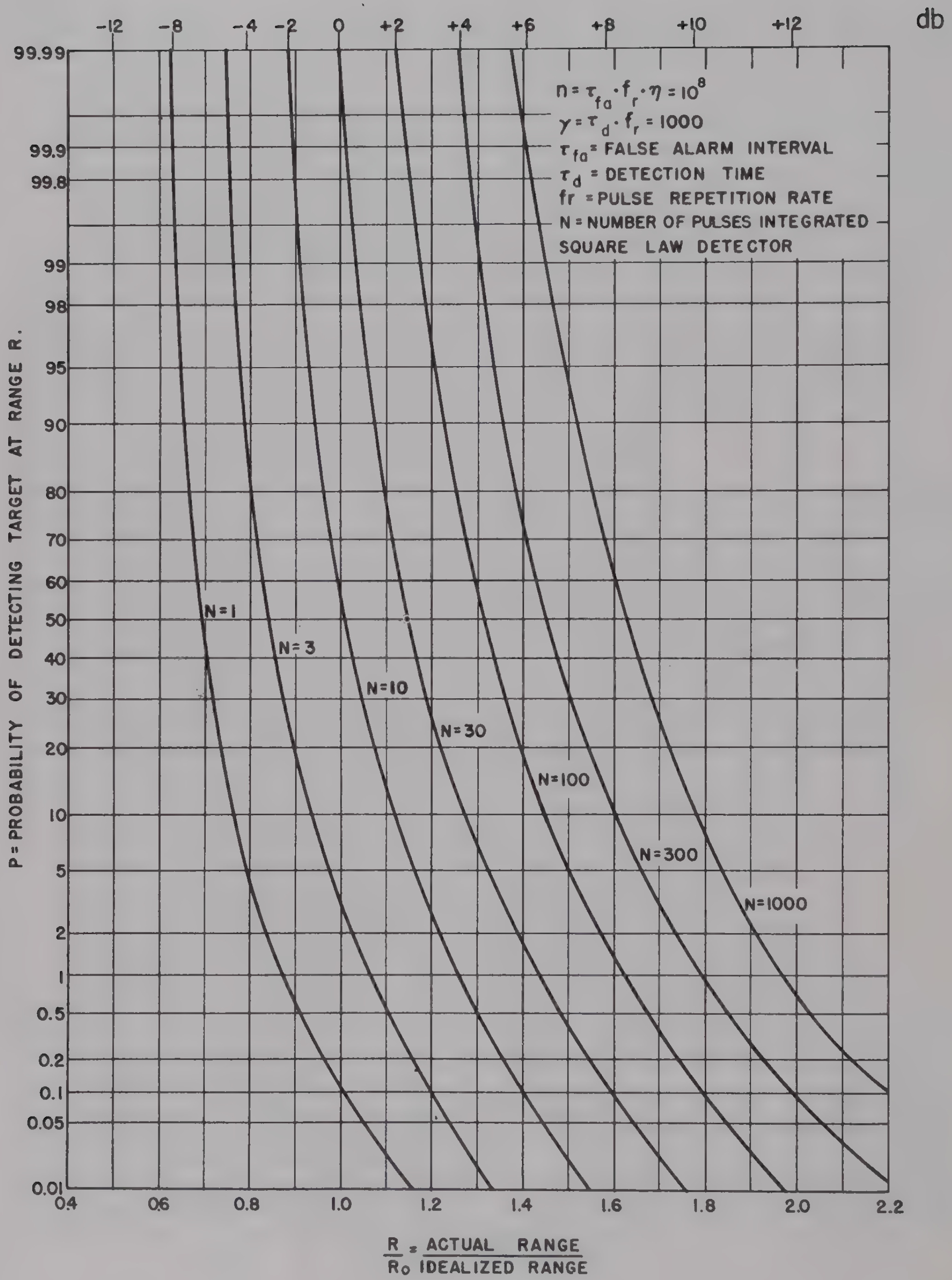

-32 -16 -12 -8 -4 -2 0 +2 +4 +6 +8 +10 db
n = τfa · fr · η = 10^8
γ = τd · fr = 100
τfa = FALSE ALARM INTERVAL
τd = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
N=1
N=3
N=10
N=30
N=100
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
99.99 99.9 99.8 99 98 95 90 80 70 60 50 40 30 20 10 5 2 1 0.5 0.2 0.1 0.05 0.01
0 0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8
R/R0 = ACTUAL RANGE / IDEALIZED RANGE

FIG. 23



db
-12 -8 -4 -2 0 +2 +4 +6 +8 +10 +12
$n = \tau_{fa} \cdot f_r \cdot \eta = 10^8$
$\gamma = \tau_d \cdot f_r = 1000$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
99.99 99.9 99.8 99 98 95 90 80 70 60 50 40 30 20 10 5 2 1 0.5 0.2 0.1 0.05 0.01
N=1
N=3
N=10
N=30
N=100
N=300
N=1000
0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0 2.2
$\frac{R}{R_0} = \frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$

FIG. 24

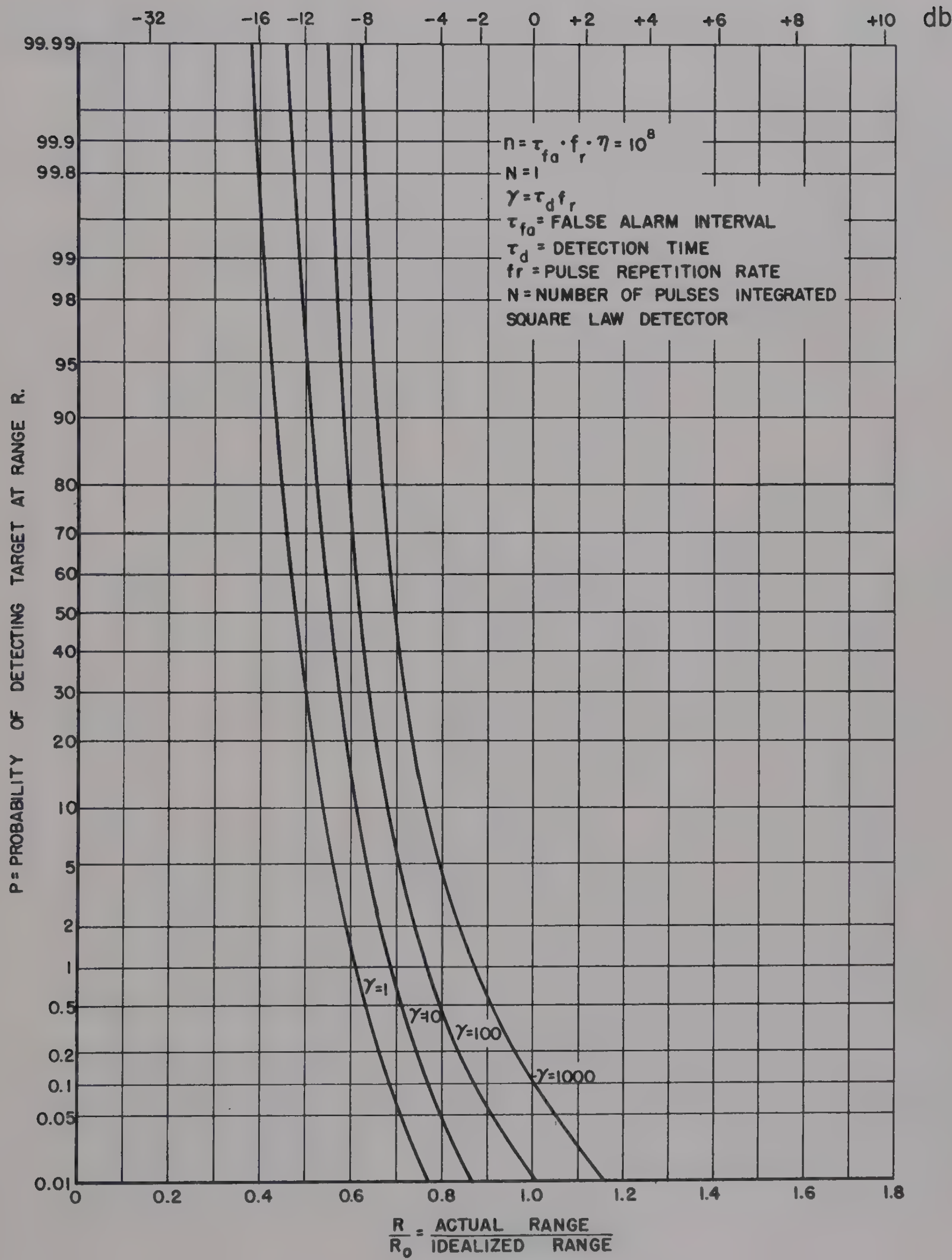

-32 -16 -12 -8 -4 -2 0 +2 +4 +6 +8 +10 db
n = τfa · fr · η = 10^8
N = 1
γ = τd fr
τfa = FALSE ALARM INTERVAL
τd = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
γ=1
γ=10
γ=100
γ=1000
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
99.99 99.9 99.8 99 98 95 90 80 70 60 50 40 30 20 10 5 2 1 0.5 0.2 0.1 0.05 0.01
0 0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8
R/R0 = ACTUAL RANGE / IDEALIZED RANGE

FIG. 25

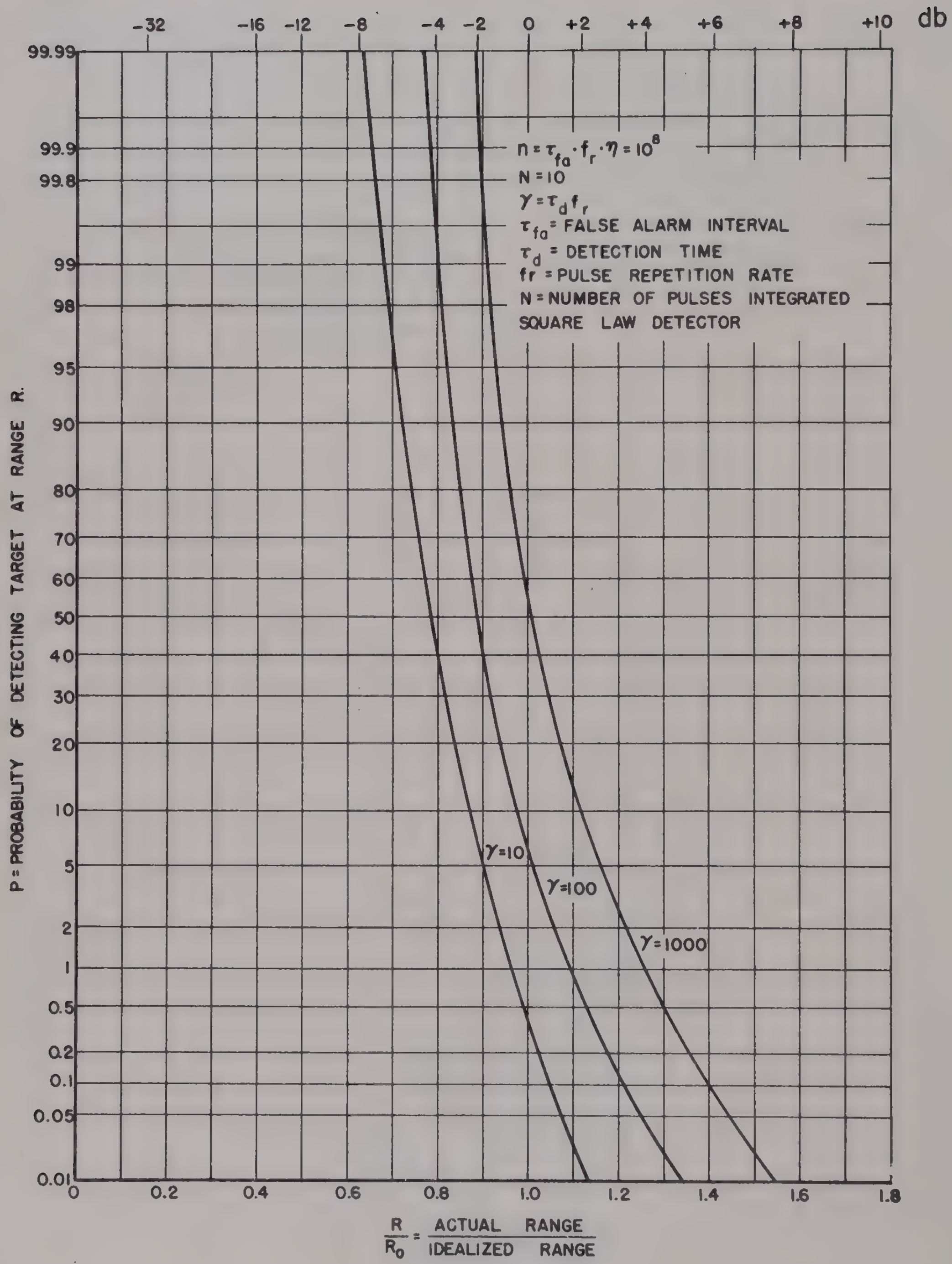
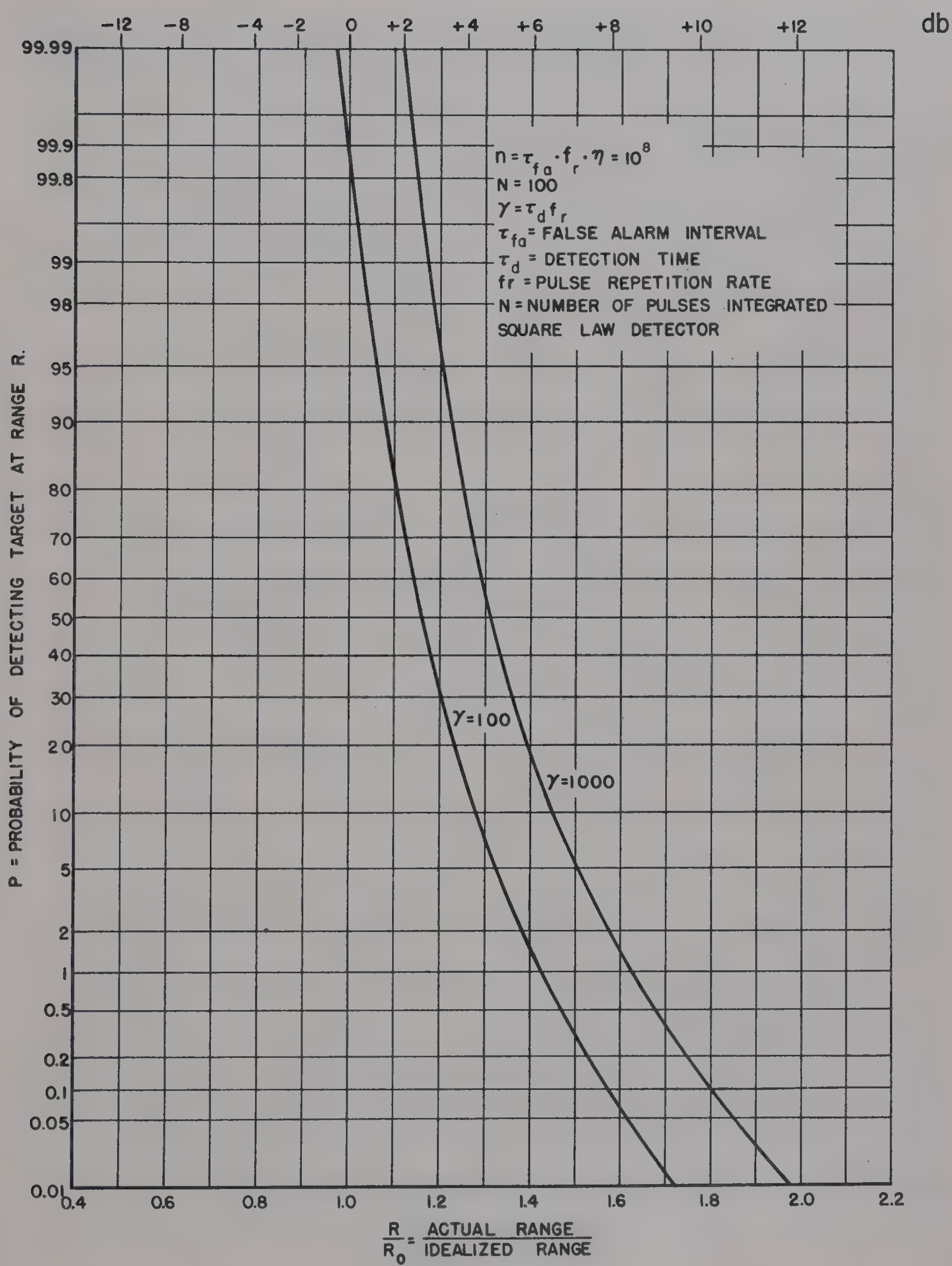
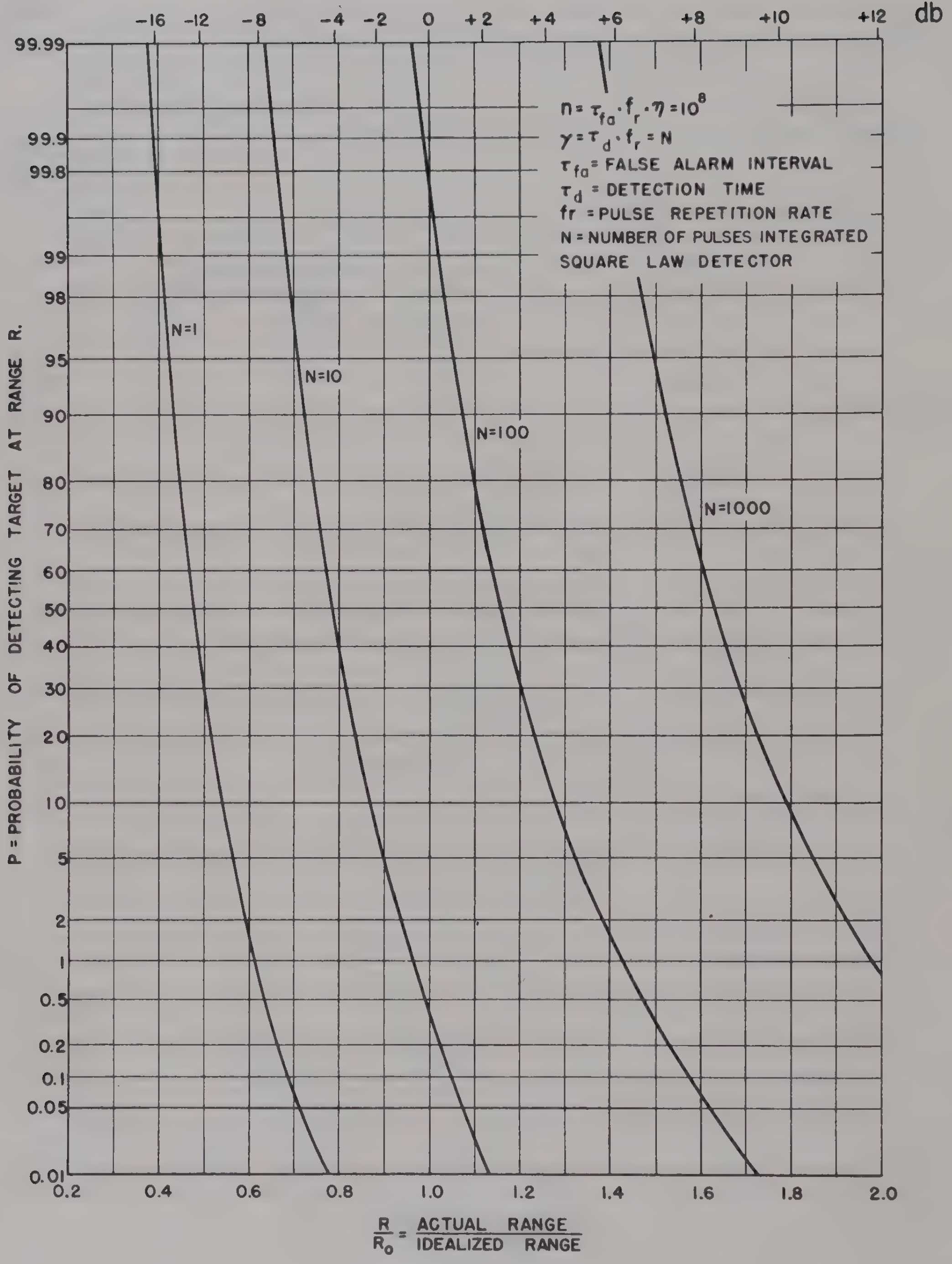

-32 -16 -12 -8 -4 -2 0 +2 +4 +6 +8 +10 db
$n = \tau_{fa} \cdot f_r \cdot \eta = 10^8$
N = 10
$\gamma = \tau_d f_r$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
γ=10
γ=100
γ=1000
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
99.99 99.9 99.8 99 98 95 90 80 70 60 50 40 30 20 10 5 2 1 0.5 0.2 0.1 0.05 0.01
0 0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8
$\frac{R}{R_0}$ = $\frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$

FIG. 26



db
-12 -8 -4 -2 0 +2 +4 +6 +8 +10 +12
$n = \tau_{fa} \cdot f_r \cdot \eta = 10^8$
N = 100
$\gamma = \tau_d f_r$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
γ=100
γ=1000
99.99 99.9 99.8 99 98 95 90 80 70 60 50 40 30 20 10 5 2 1 0.5 0.2 0.1 0.05 0.01
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0 2.2
$\frac{R}{R_0} = \frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$

FIG. 27



db
-16 -12 -8 -4 -2 0 +2 +4 +6 +8 +10 +12
99.99
99.9
99.8
99
98
95
90
80
70
60
50
40
30
20
10
5
2
1
0.5
0.2
0.1
0.05
0.01
0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
$n = \tau_{fa} \cdot f_r \cdot \eta = 10^8$
$\gamma = \tau_d \cdot f_r = N$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
N=1
N=10
N=100
N=1000
$\frac{R}{R_0} = \frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$

FIG. 28

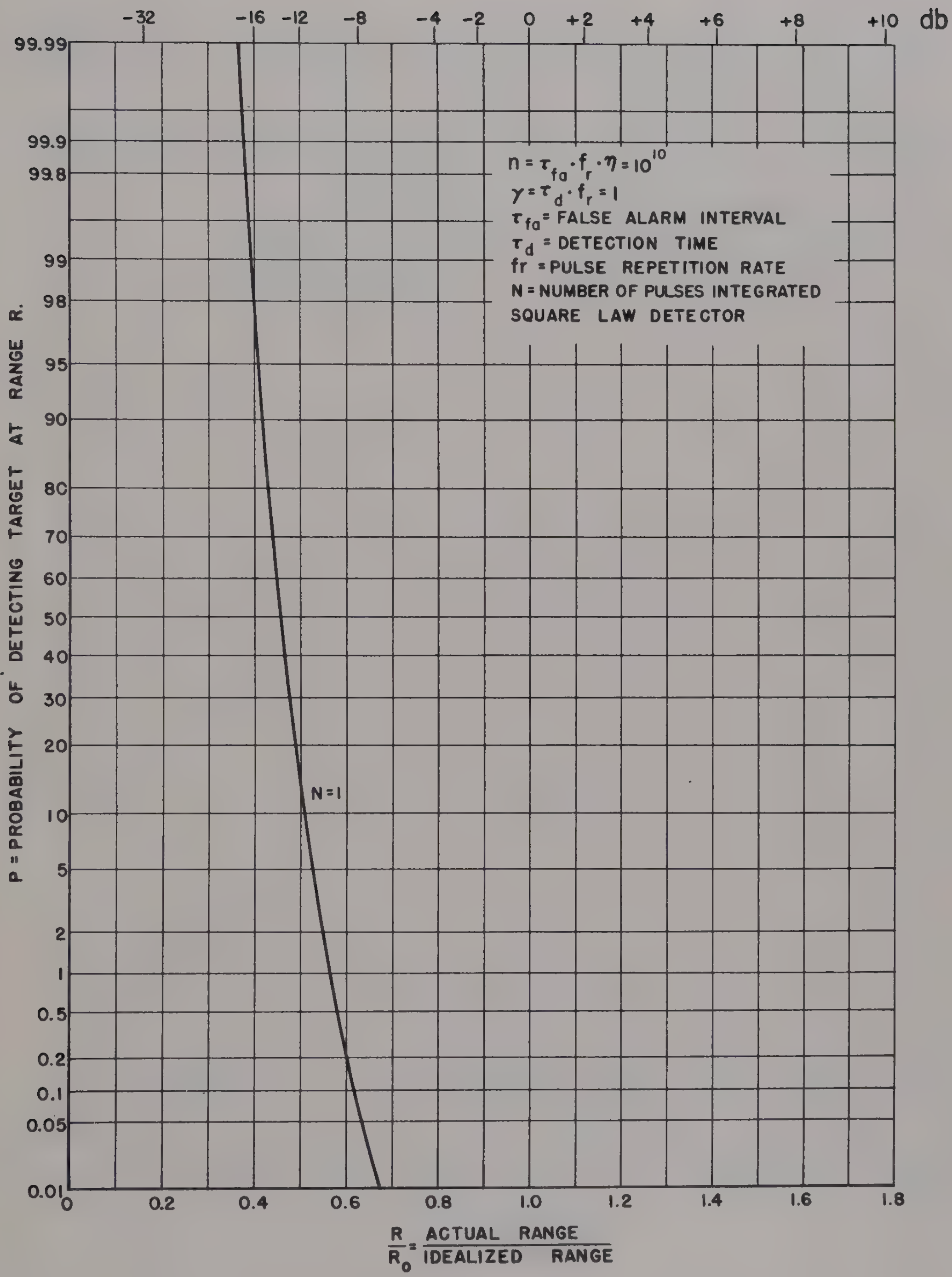

-32 -16 -12 -8 -4 -2 0 +2 +4 +6 +8 +10 db
$n = \tau_{fa} \cdot f_r \cdot \eta = 10^{10}$
$\gamma = \tau_d \cdot f_r = 1$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
N=1
99.99 99.9 99.8 99 98 95 90 80 70 60 50 40 30 20 10 5 2 1 0.5 0.2 0.1 0.05 0.01
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
0 0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8
$\frac{R}{R_0} = \frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$

FIG. 29

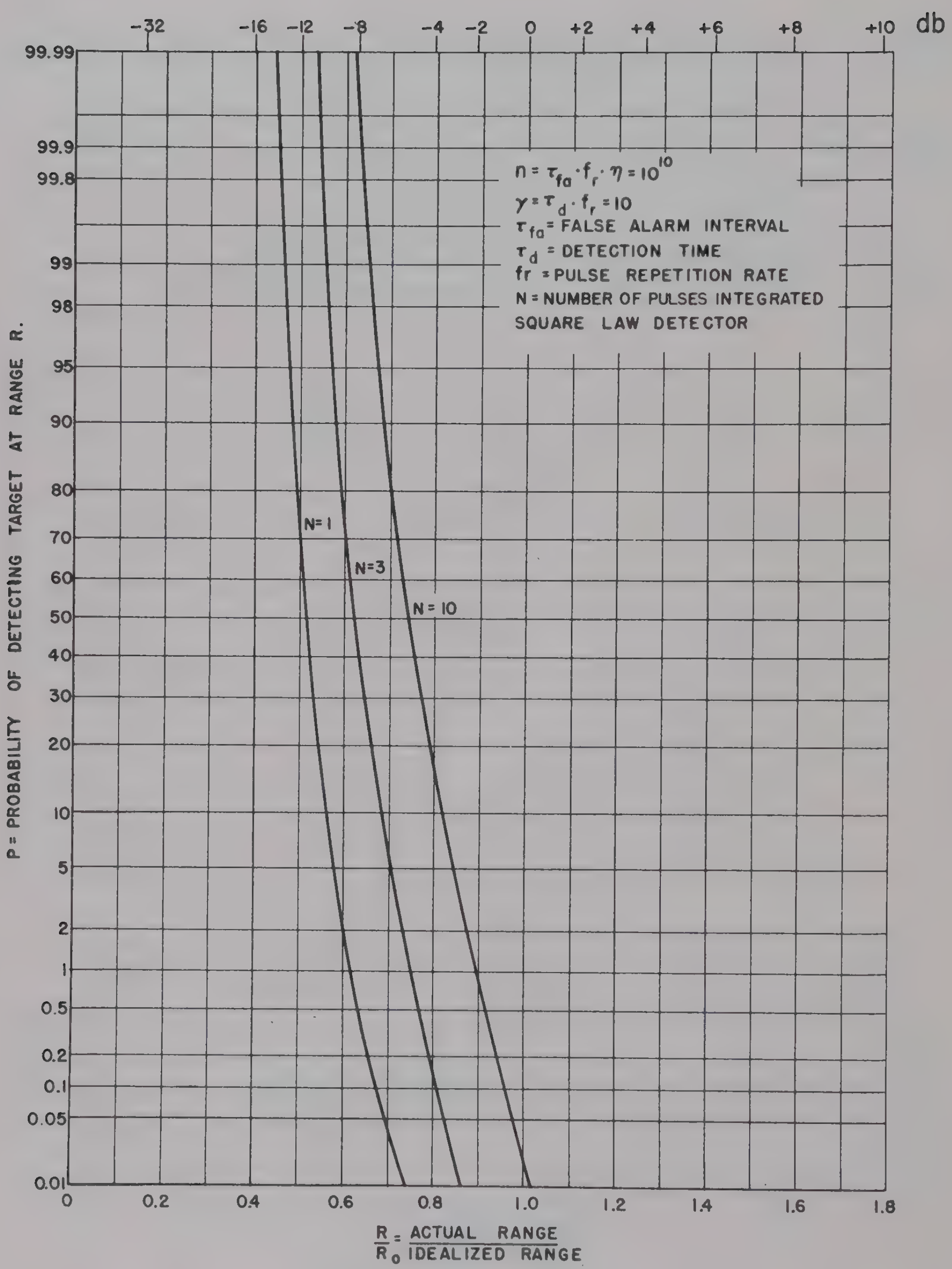

-32 -16 -12 -8 -4 -2 0 +2 +4 +6 +8 +10 db
n = $\tau_{fa} \cdot f_r \cdot \eta = 10^{10}$
$\gamma = \tau_d \cdot f_r = 10$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
N= 1
N=3
N = 10
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
99.99 99.9 99.8 99 98 95 90 80 70 60 50 40 30 20 10 5 2 1 0.5 0.2 0.1 0.05 0.01
0 0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8
$\frac{R}{R_0} = \frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$

FIG. 30

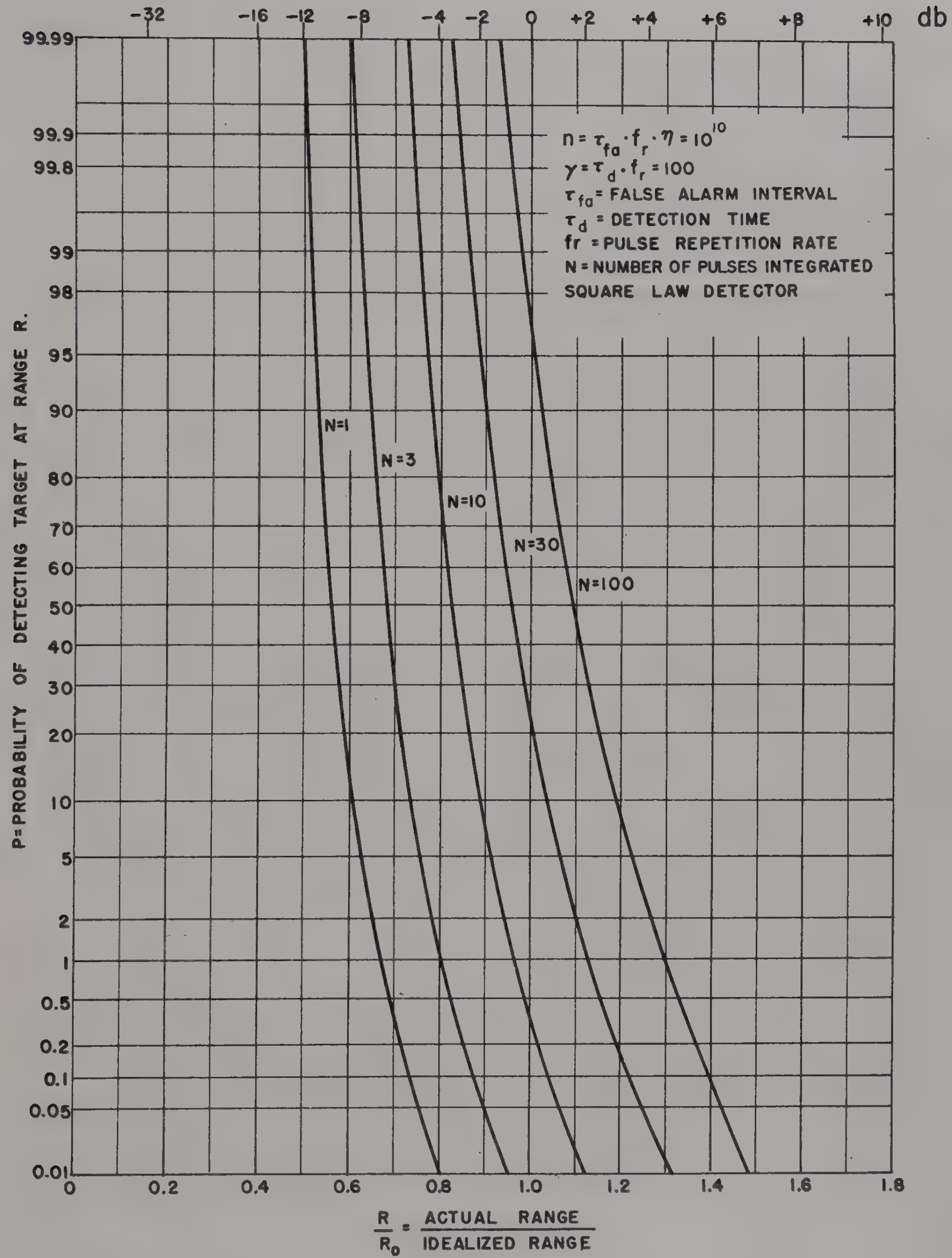
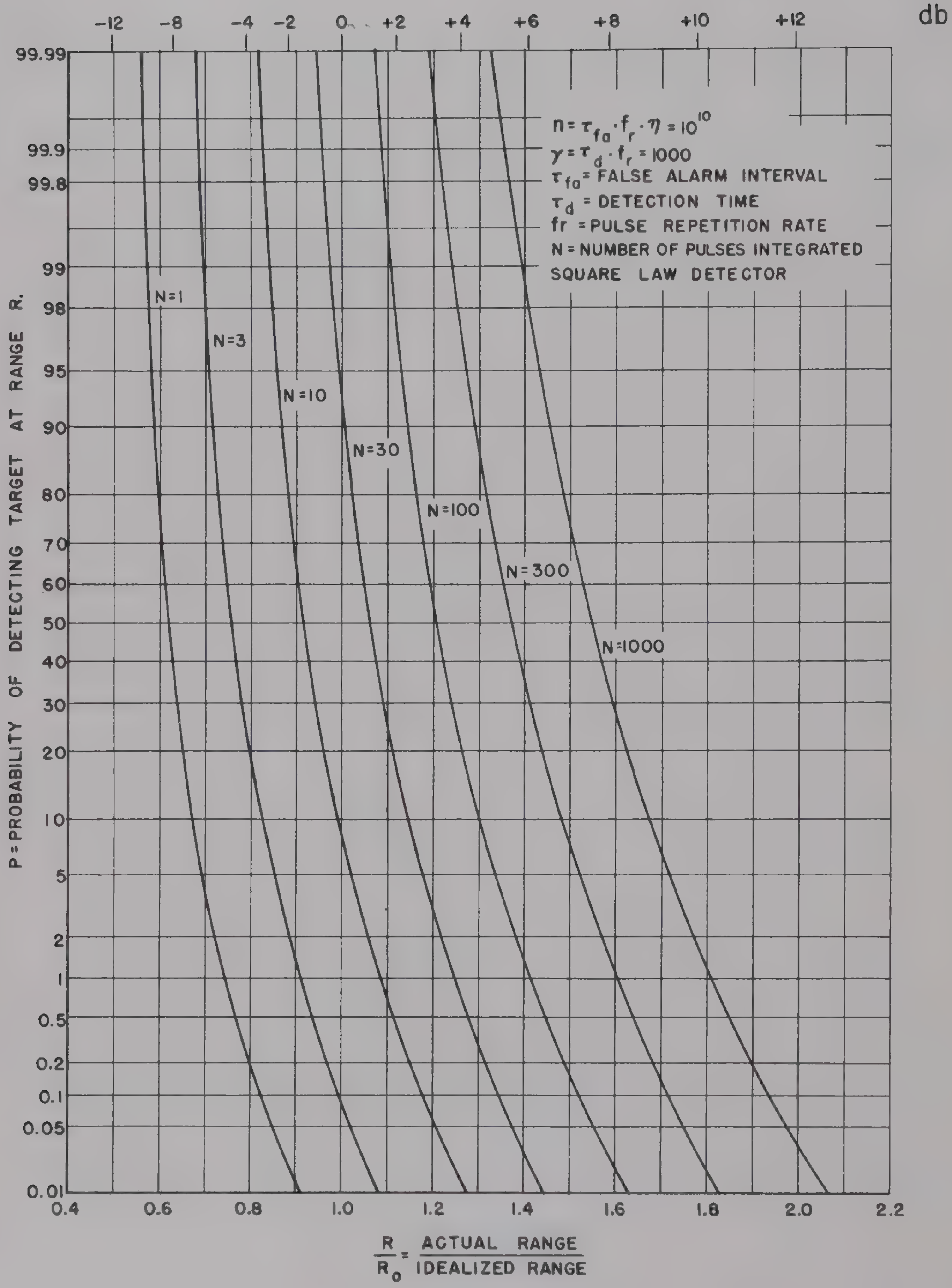
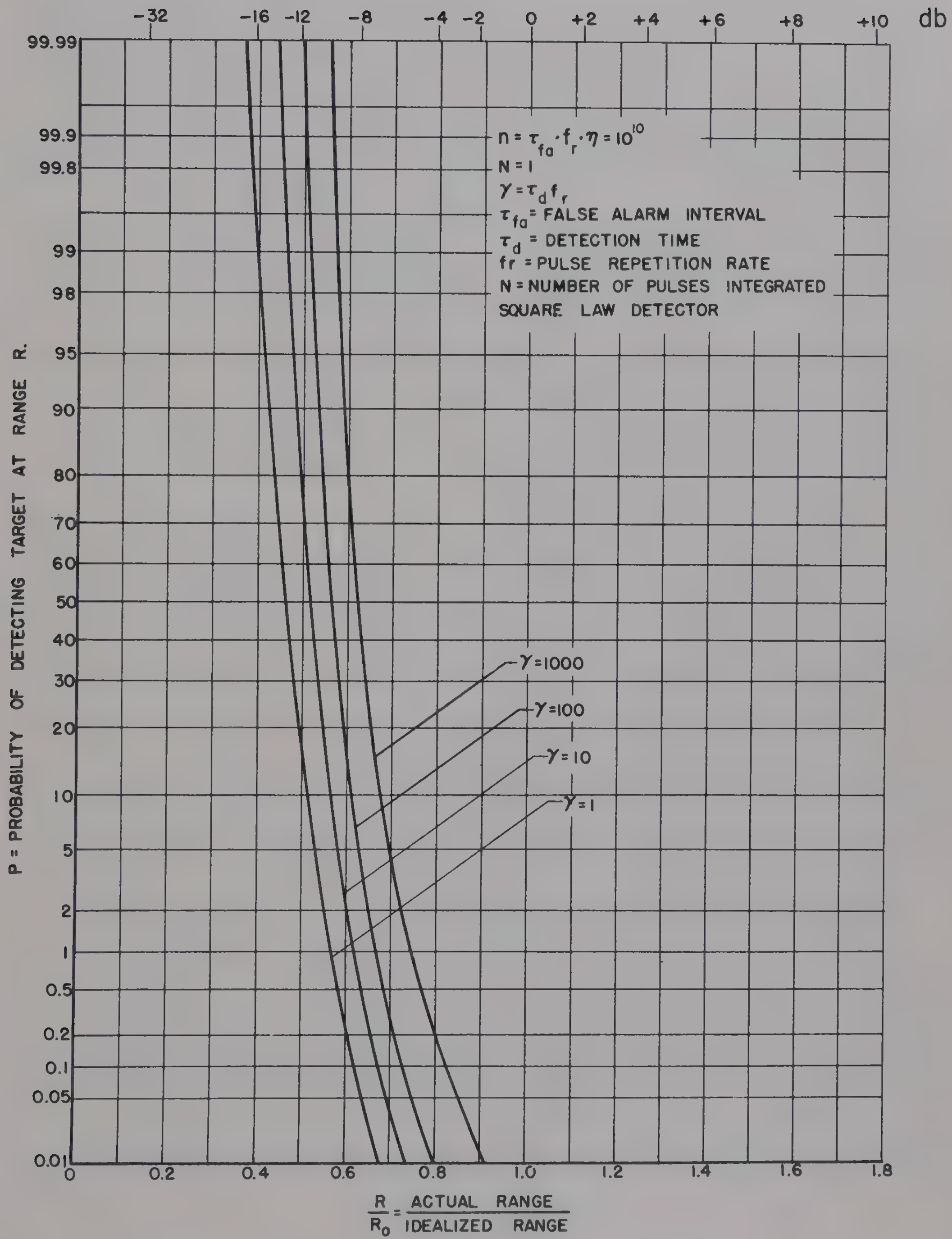

-32 -16 -12 -8 -4 -2 0 +2 +4 +6 +8 +10 db
$n = \tau_{fa} \cdot f_r \cdot \eta = 10^{10}$
$\gamma = \tau_d \cdot f_r = 100$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
N=1
N=3
N=10
N=30
N=100
P=PROBABILITY OF DETECTING TARGET AT RANGE R.
99.99 99.9 99.8 99 98 95 90 80 70 60 50 40 30 20 10 5 2 1 0.5 0.2 0.1 0.05 0.01
0 0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8
$\frac{R}{R_0} = \frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$

FIG. 31



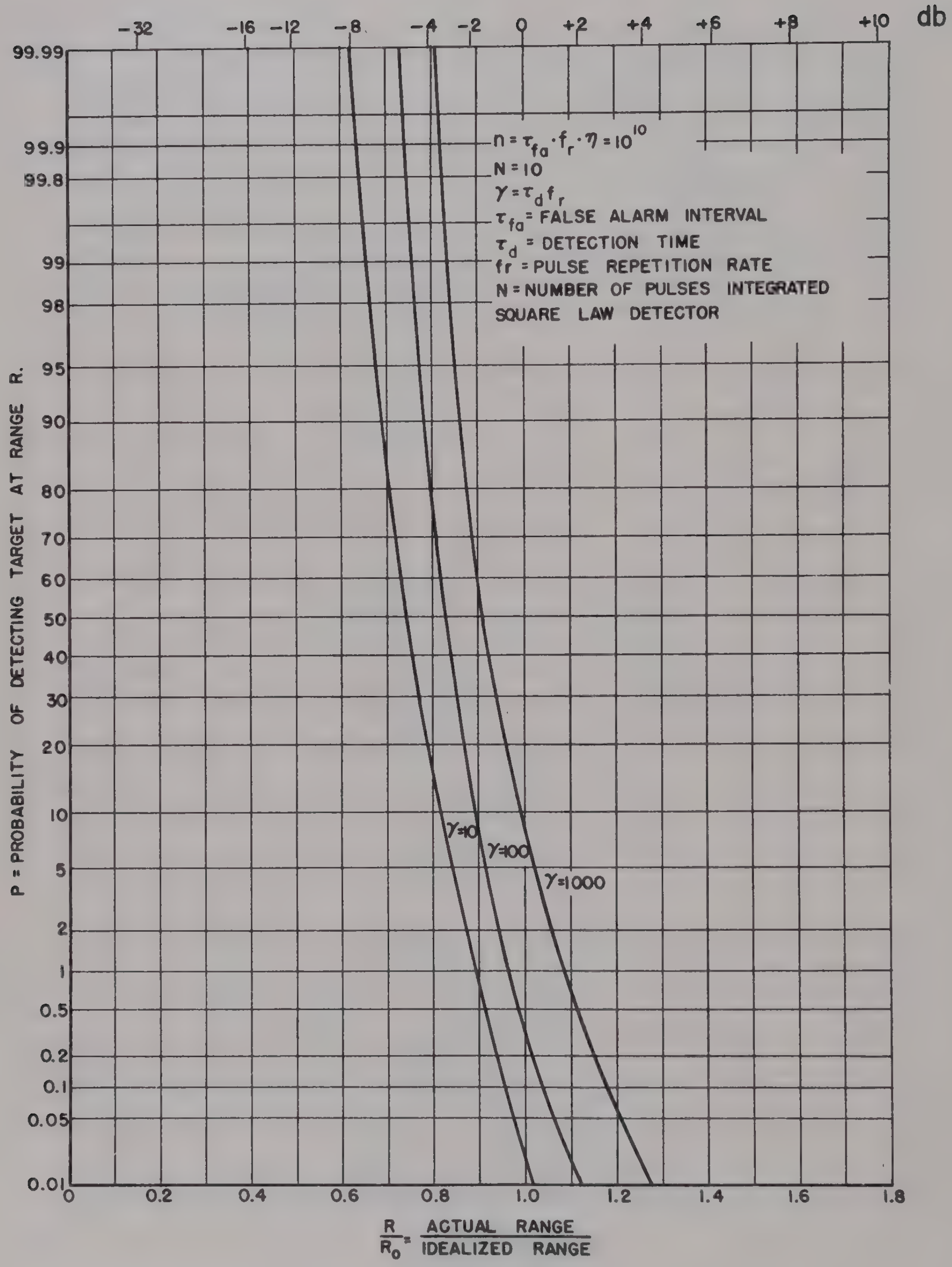
db
-12 -8 -4 -2 0 +2 +4 +6 +8 +10 +12
$n = \tau_{fa} \cdot f_r \cdot \eta = 10^{10}$
$\gamma = \tau_d \cdot f_r = 1000$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
N=1
N=3
N=10
N=30
N=100
N=300
N=1000
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
99.99 99.9 99.8 99 98 95 90 80 70 60 50 40 30 20 10 5 2 1 0.5 0.2 0.1 0.05 0.01
0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0 2.2
$\frac{R}{R_0} = \frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$

FIG. 32



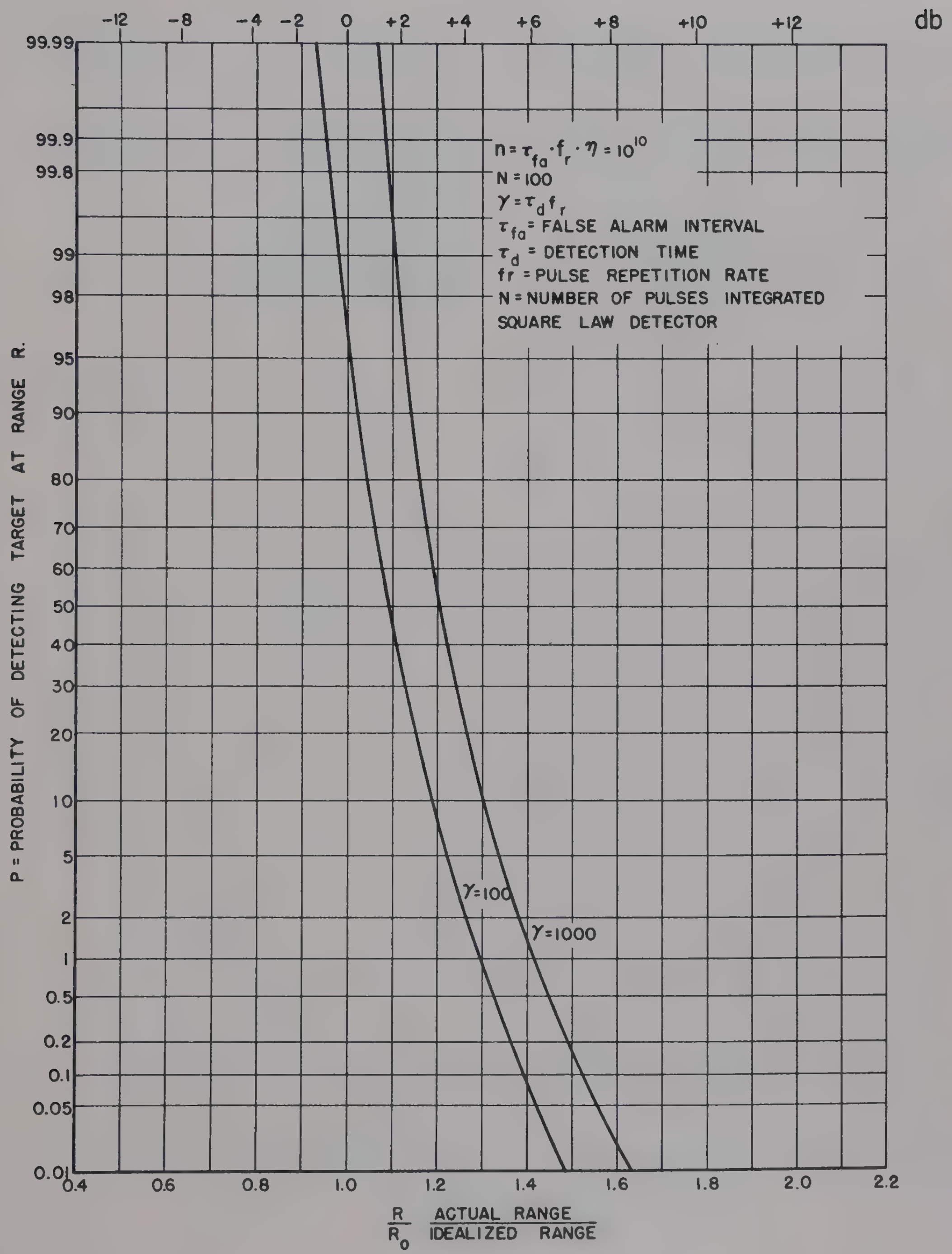
db
-32
-16
-12
-8
-4
-2
0
+2
+4
+6
+8
+10
99.99
99.9
99.8
99
98
95
90
80
70
60
50
40
30
20
10
5
2
1
0.5
0.2
0.1
0.05
0.01
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
$n = \tau_{fa} \cdot f_r \cdot \eta = 10^{10}$
N = 1
$\gamma = \tau_d f_r$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
γ = 1000
γ = 100
γ = 10
γ = 1
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
$\frac{R}{R_o} = \frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$

FIG. 33



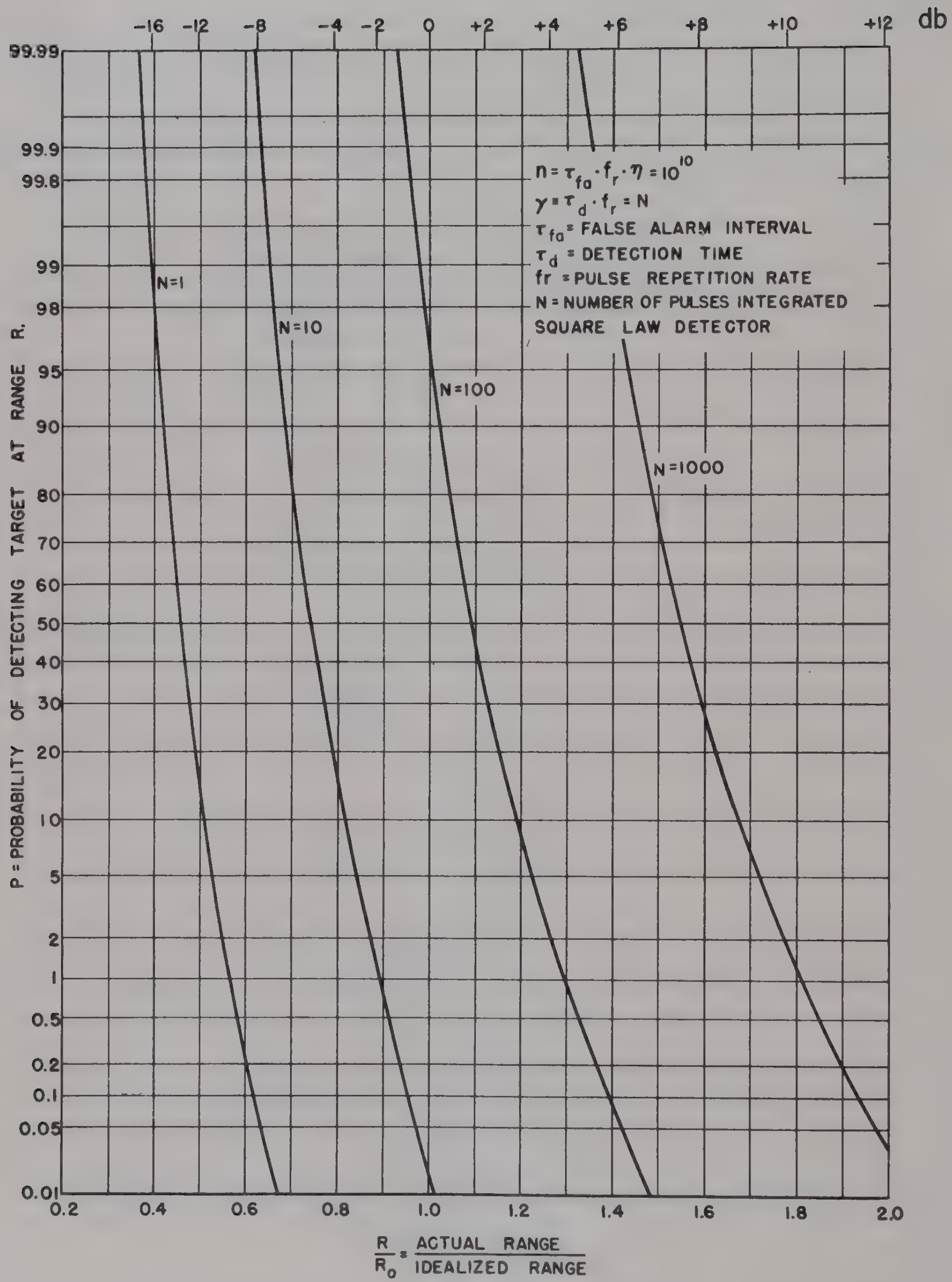
db
-32
-16
-12
-8
-4
-2
0
+2
+4
+6
+8
+10
99.99
99.9
99.8
99
98
95
90
80
70
60
50
40
30
20
10
5
2
1
0.5
0.2
0.1
0.05
0.01
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
$n = \tau_{fa} \cdot f_r \cdot \eta = 10^{10}$
N = 10
$\gamma = \tau_d f_r$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
γ=10
γ=100
γ=1000
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
$\frac{R}{R_0} = \frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$

FIG. 34



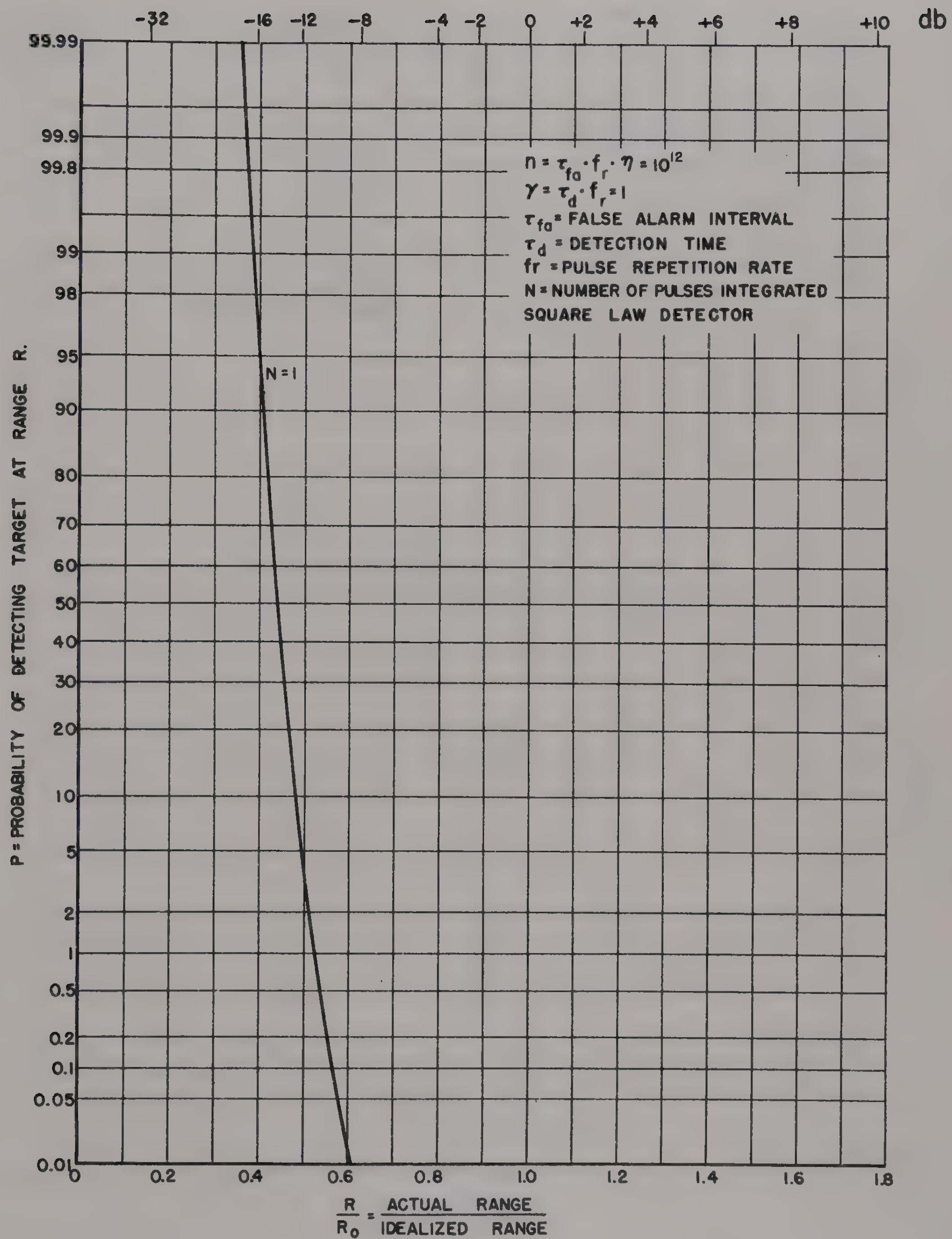
db
−12 −8 −4 −2 0 +2 +4 +6 +8 +10 +12
99.99 99.9 99.8 99 98 95 90 80 70 60 50 40 30 20 10 5 2 1 0.5 0.2 0.1 0.05 0.01
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
$n = \tau_{fa} \cdot f_r \cdot \eta = 10^{10}$
N = 100
$\gamma = \tau_d f_r$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
γ=100
γ=1000
0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0 2.2
$\frac{R}{R_0}$ $\frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$

FIG. 35



-16 -12 -8 -4 -2 0 +2 +4 +6 +8 +10 +12 db
99.99
99.9
99.8
99
98
95
90
80
70
60
50
40
30
20
10
5
2
1
0.5
0.2
0.1
0.05
0.01
0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
$\frac{R}{R_0}$ = $\frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$
$n = \tau_{fa} \cdot f_r \cdot \eta = 10^{10}$
$\gamma = \tau_d \cdot f_r = N$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
N=1
N=10
N=100
N=1000

FIG. 36



db
-32 -16 -12 -8 -4 -2 0 +2 +4 +6 +8 +10
99.99 99.9 99.8 99 98 95 90 80 70 60 50 40 30 20 10 5 2 1 0.5 0.2 0.1 0.05 0.01
0 0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
n = $\tau_{fa} \cdot f_r \cdot \eta = 10^{12}$
$\gamma = \tau_d \cdot f_r = 1$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
N = 1
$\frac{R}{R_0}$ = $\frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$

FIG. 37

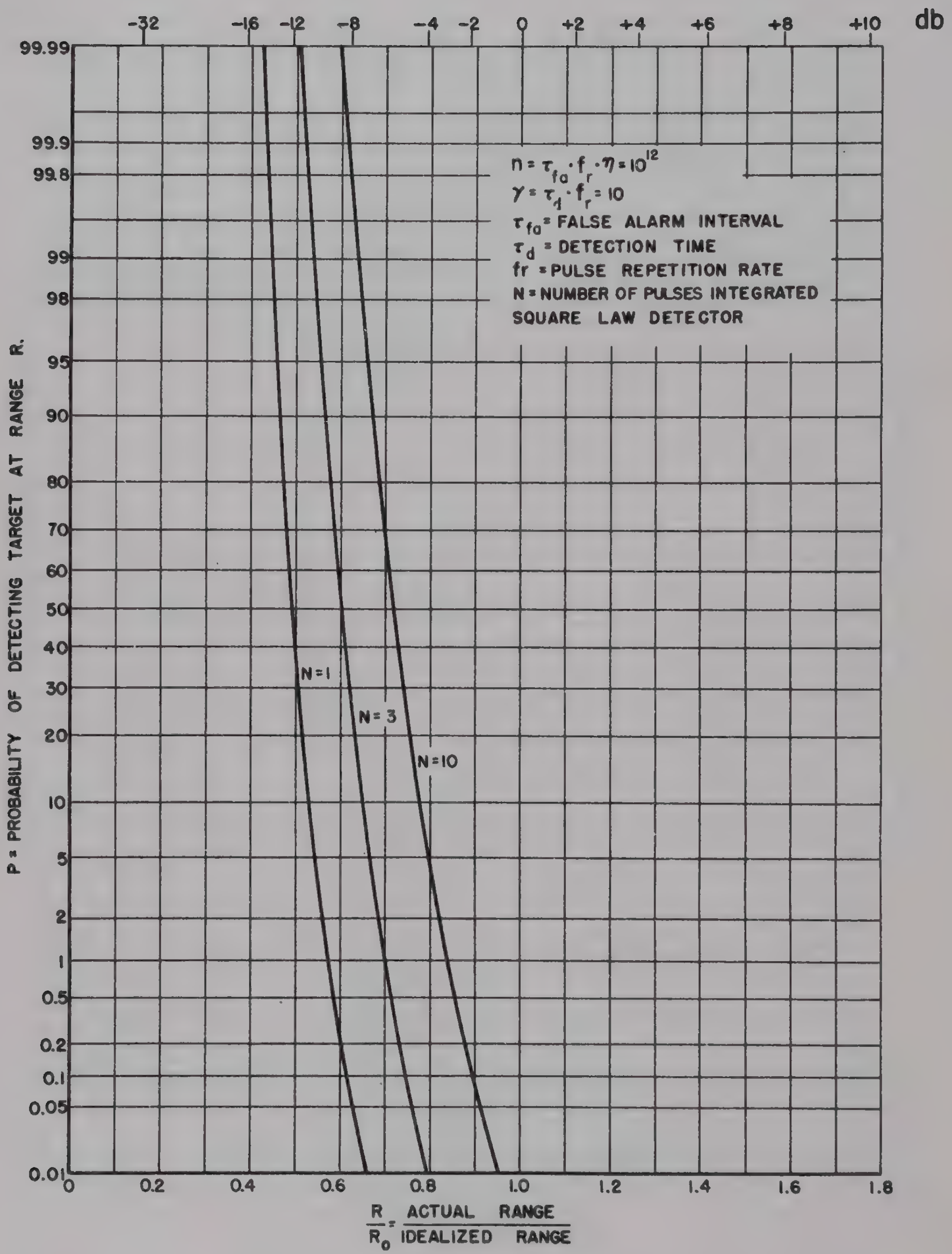
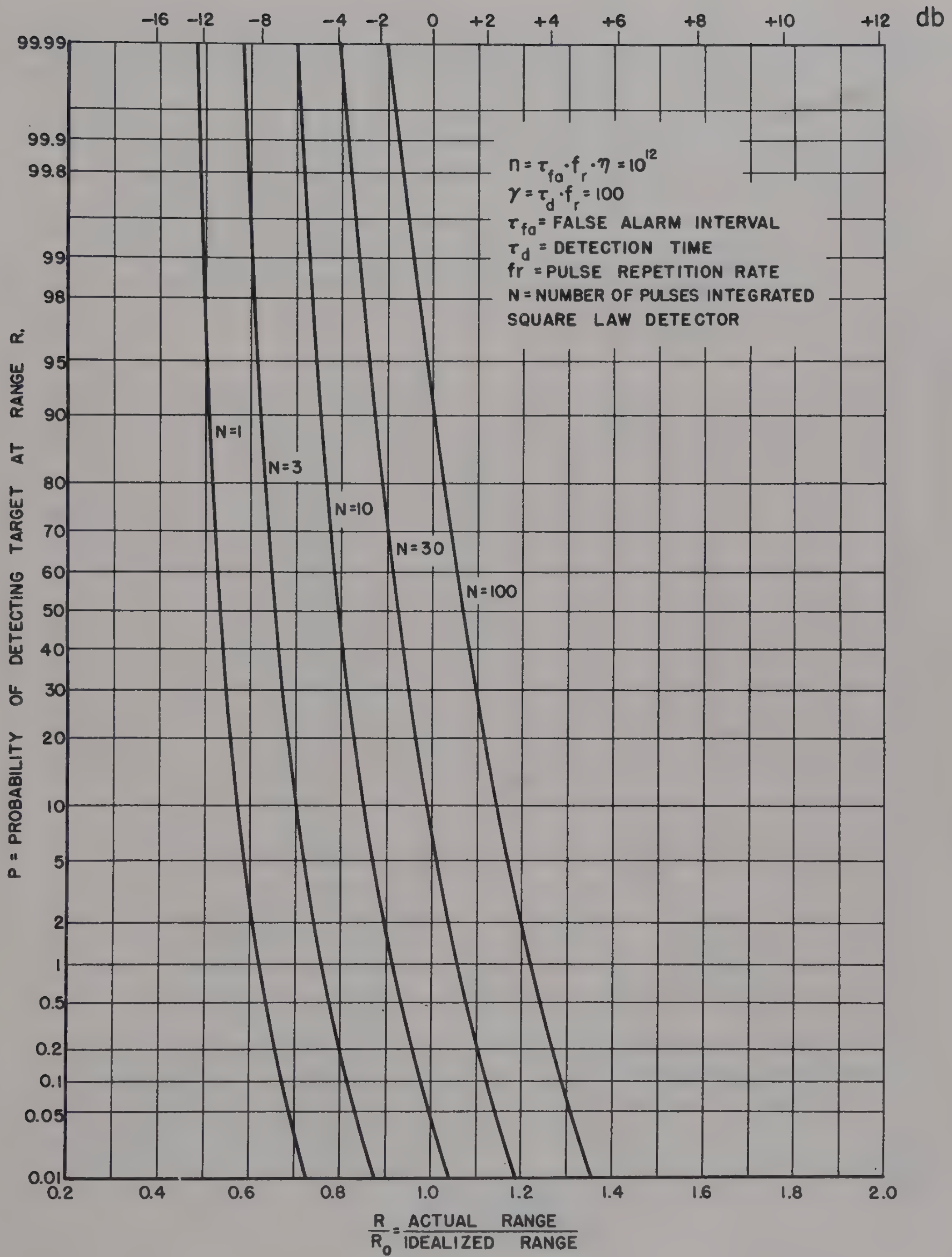
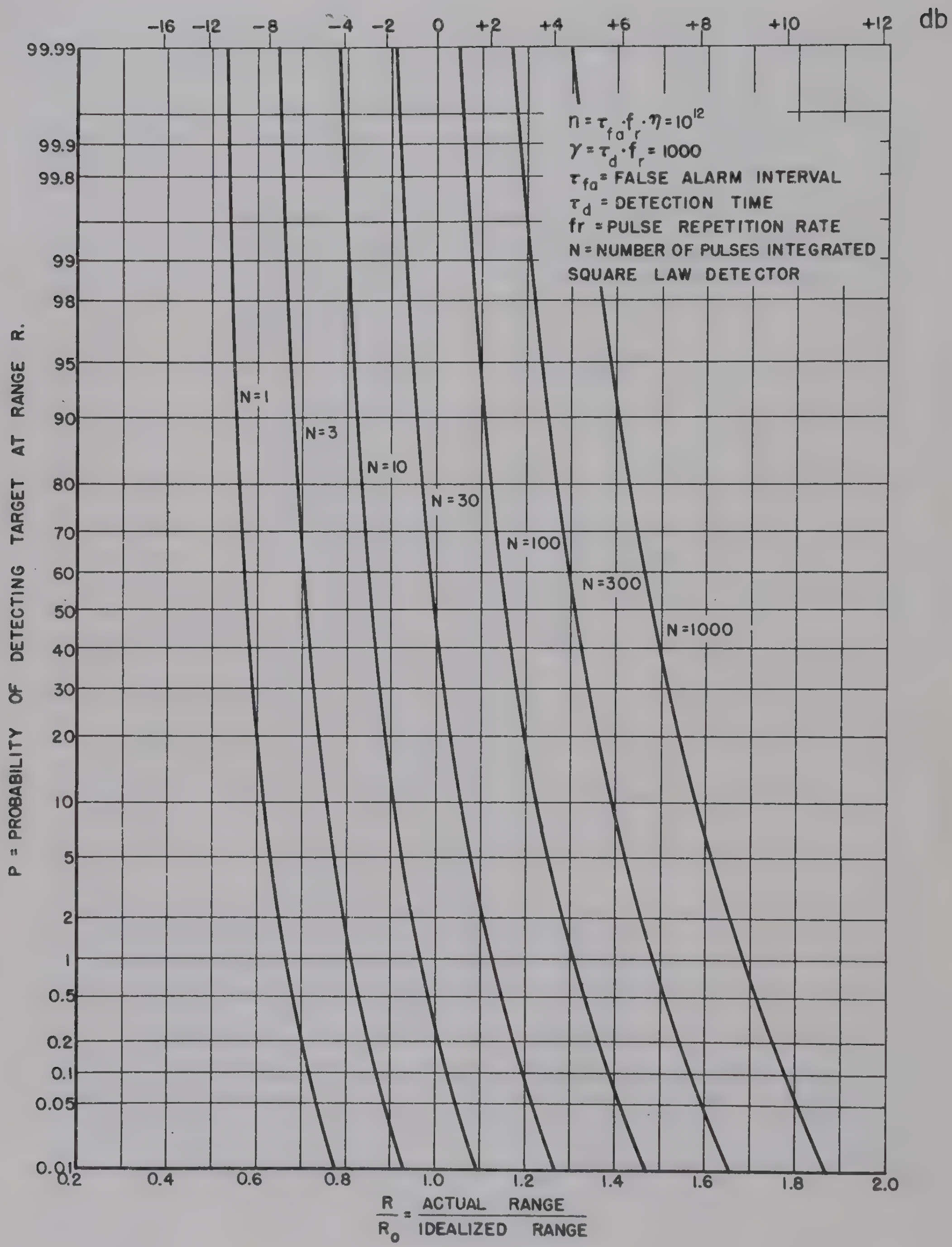

db
-32 -16 -12 -8 -4 -2 0 +2 +4 +6 +8 +10
99.99 99.9 99.8 99 98 95 90 80 70 60 50 40 30 20 10 5 2 1 0.5 0.2 0.1 0.05 0.01
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
n = $\tau_{fa} \cdot f_r \cdot \eta = 10^{12}$
$\gamma = \tau_d \cdot f_r = 10$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
N=1
N=3
N=10
0 0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8
$\frac{R}{R_0} = \frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$

FIG. 38



db
-16
-12
-8
-4
-2
0
+2
+4
+6
+8
+10
+12
99.99
99.9
99.8
99
98
95
90
80
70
60
50
40
30
20
10
5
2
1
0.5
0.2
0.1
0.05
0.01
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
$n = \tau_{fa} \cdot f_r \cdot \eta = 10^{12}$
$\gamma = \tau_d \cdot f_r = 100$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
N=1
N=3
N=10
N=30
N=100
$\frac{R}{R_0} = \frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$

FIG. 39



db
-16 -12 -8 -4 -2 0 +2 +4 +6 +8 +10 +12
$n = \tau_{fa} \cdot f_r \cdot \eta = 10^{12}$
$\gamma = \tau_d \cdot f_r = 1000$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
N=1
N=3
N=10
N=30
N=100
N=300
N=1000
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
99.99 99.9 99.8 99 98 95 90 80 70 60 50 40 30 20 10 5 2 1 0.5 0.2 0.1 0.05 0.01
0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0
$\frac{R}{R_o} = \frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$

FIG. 40

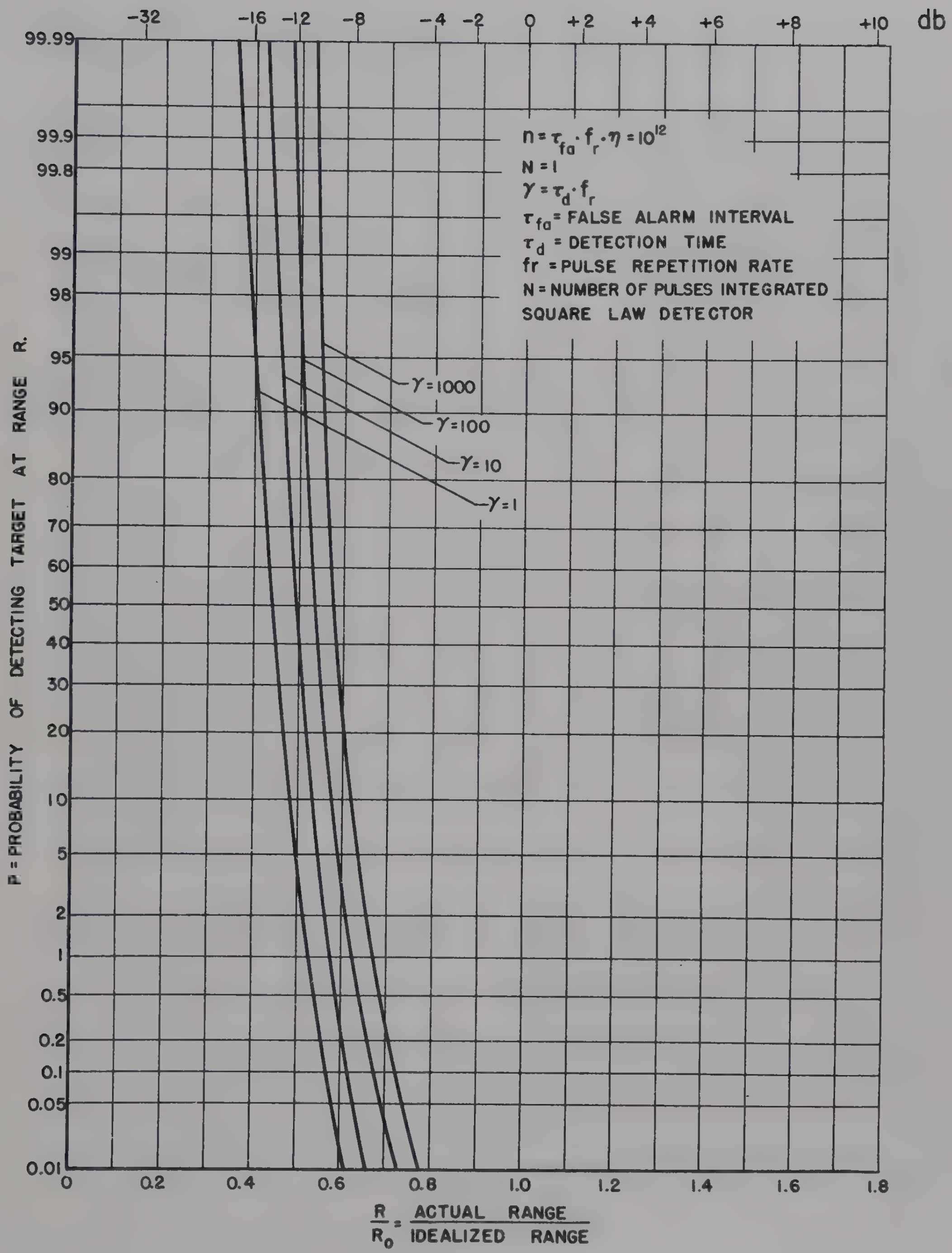
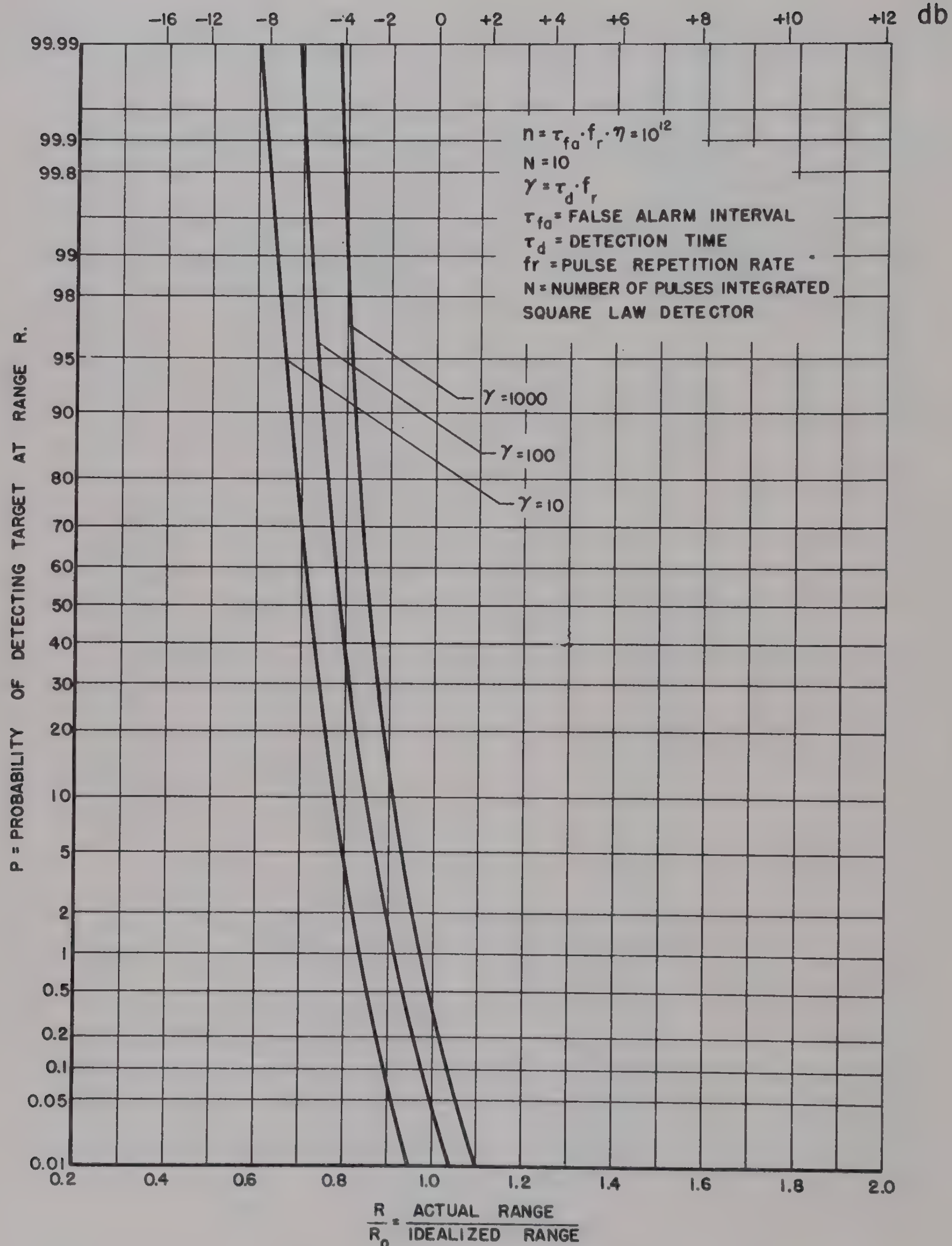
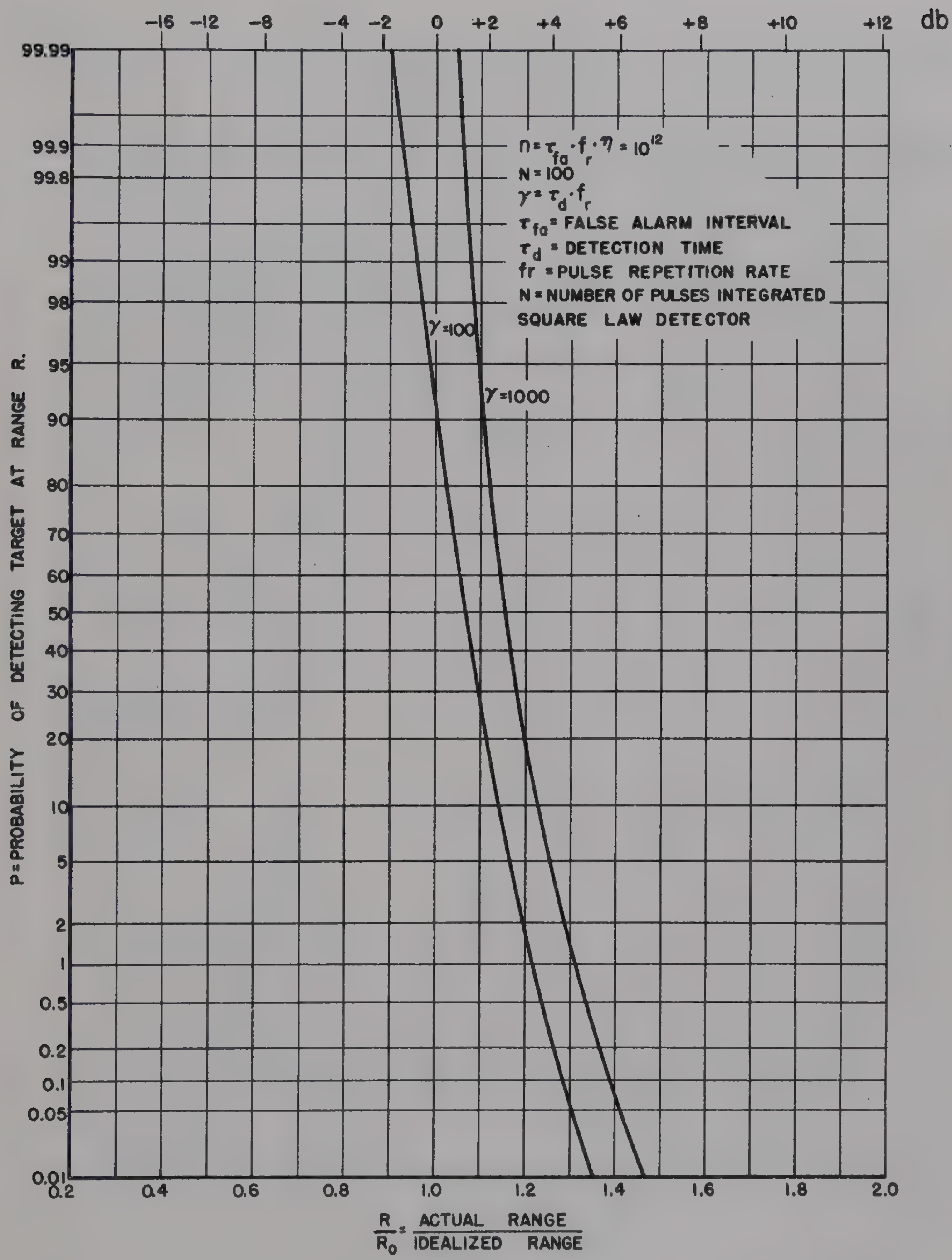

$n = \tau_{fa} \cdot f_r \cdot \eta = 10^{12}$
N = 1
$\gamma = \tau_d \cdot f_r$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
$\gamma = 1000$
$\gamma = 100$
$\gamma = 10$
$\gamma = 1$
db
-32 -16 -12 -8 -4 -2 0 +2 +4 +6 +8 +10
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
99.99 99.9 99.8 99 98 95 90 80 70 60 50 40 30 20 10 5 2 1 0.5 0.2 0.1 0.05 0.01
0 0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8
$\frac{R}{R_0} = \frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$

FIG. 41



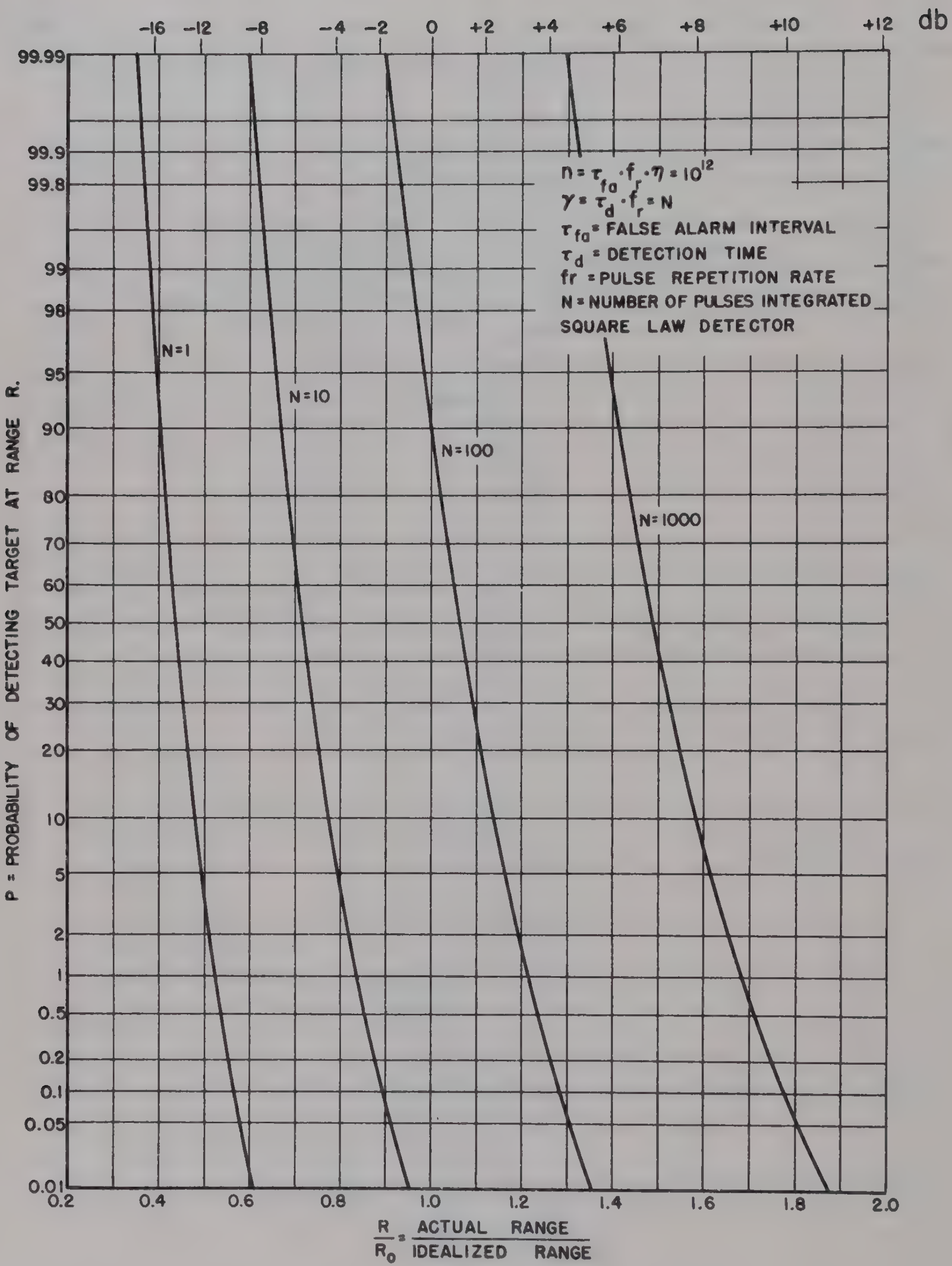
db
−16 −12 −8 −4 −2 0 +2 +4 +6 +8 +10 +12
$n = \tau_{fa} \cdot f_r \cdot \eta = 10^{12}$
N = 10
$\gamma = \tau_d \cdot f_r$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
γ = 1000
γ = 100
γ = 10
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
99.99 99.9 99.8 99 98 95 90 80 70 60 50 40 30 20 10 5 2 1 0.5 0.2 0.1 0.05 0.01
0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0
$\frac{R}{R_0} = \frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$

FIG. 42



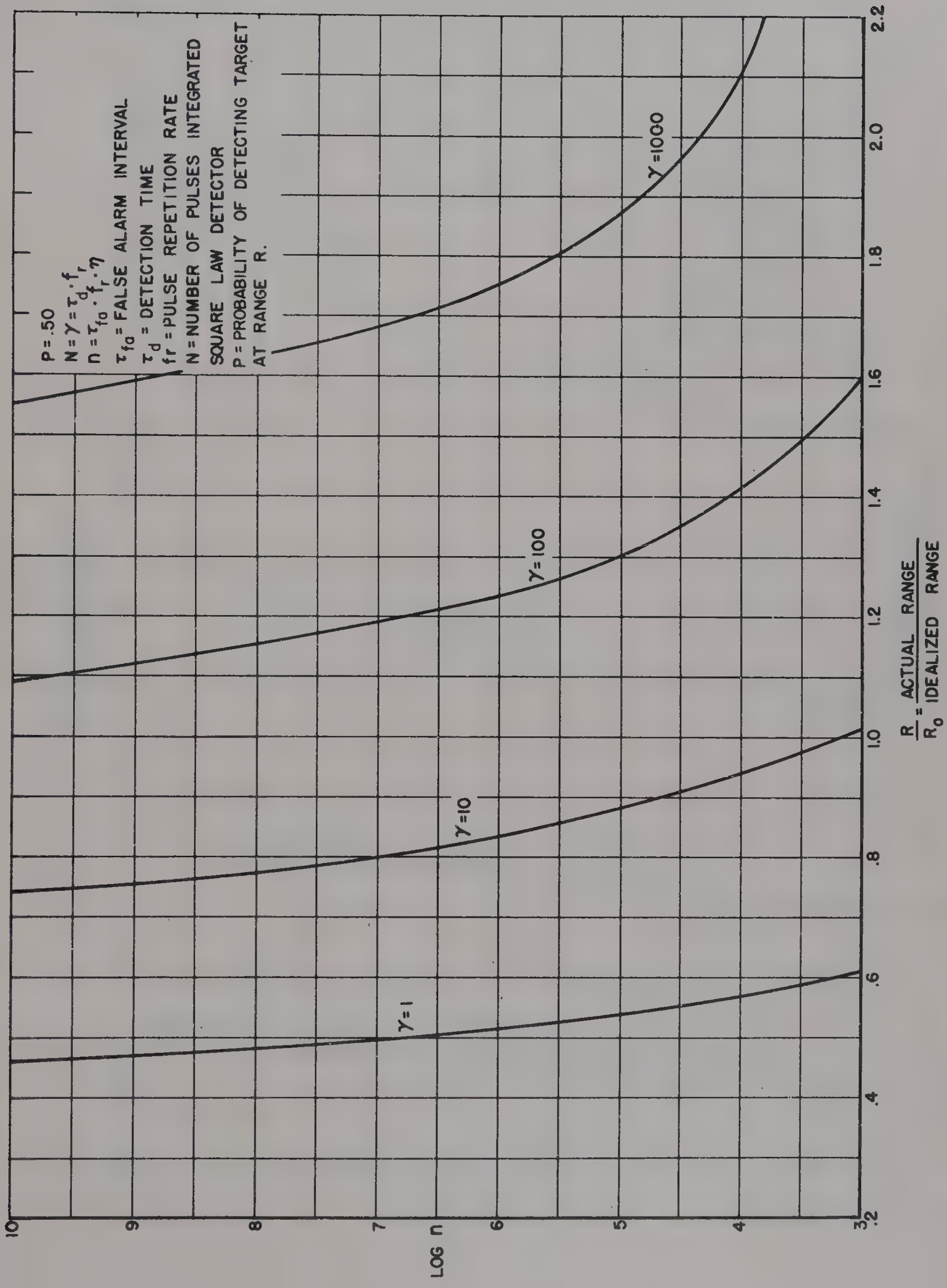
-16 -12 -8 -4 -2 0 +2 +4 +6 +8 +10 +12 db
99.99
99.9
99.8
99
98
95
90
80
70
60
50
40
30
20
10
5
2
1
0.5
0.2
0.1
0.05
0.01
0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0
$n = \tau_{fa} \cdot f_r \cdot \eta = 10^{12}$
N = 100
$\gamma = \tau_d \cdot f_r$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
γ=100
γ=1000
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
$\frac{R}{R_0} = \frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$

FIG. 43



-16
-12
-8
-4
-2
0
+2
+4
+6
+8
+10
+12
db
99.99
99.9
99.8
99
98
95
90
80
70
60
50
40
30
20
10
5
2
1
0.5
0.2
0.1
0.05
0.01
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
N=1
N=10
N=100
N=1000
$n = \tau_{fa} \cdot f_r \cdot \eta = 10^{12}$
$\gamma = \tau_d \cdot f_r = N$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
$\frac{R}{R_0} = \frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$

FIG. 44



P = .50
$N = \gamma = \tau_d \cdot f_r$
$n = \tau_{fa} \cdot f_r \cdot \eta$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
f_r = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
γ=1
γ=10
γ=100
γ=1000
LOG n
10
9
8
7
6
5
4
3
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
$\frac{R}{R_o} = \frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$

FIG. 45

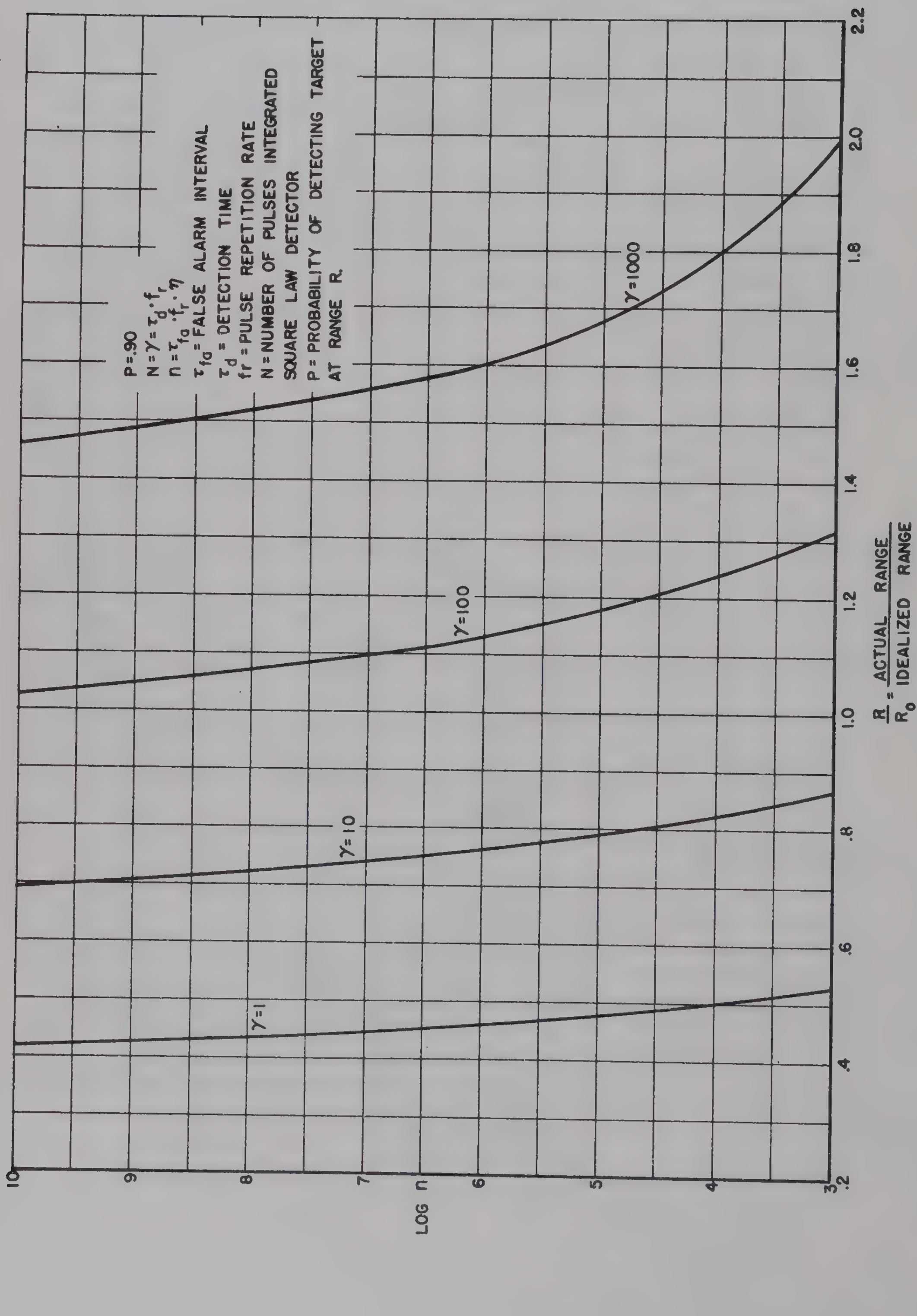

P=.90
N=γ=τd·fr
n=τfa·fr·η
τfa=FALSE ALARM INTERVAL
τd=DETECTION TIME
fr=PULSE REPETITION RATE
N=NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
P=PROBABILITY OF DETECTING TARGET
AT RANGE R.
γ=1
γ=10
γ=100
γ=1000
LOG n
10
9
8
7
6
5
4
3
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
R/R0 = ACTUAL RANGE / IDEALIZED RANGE

FIG. 46

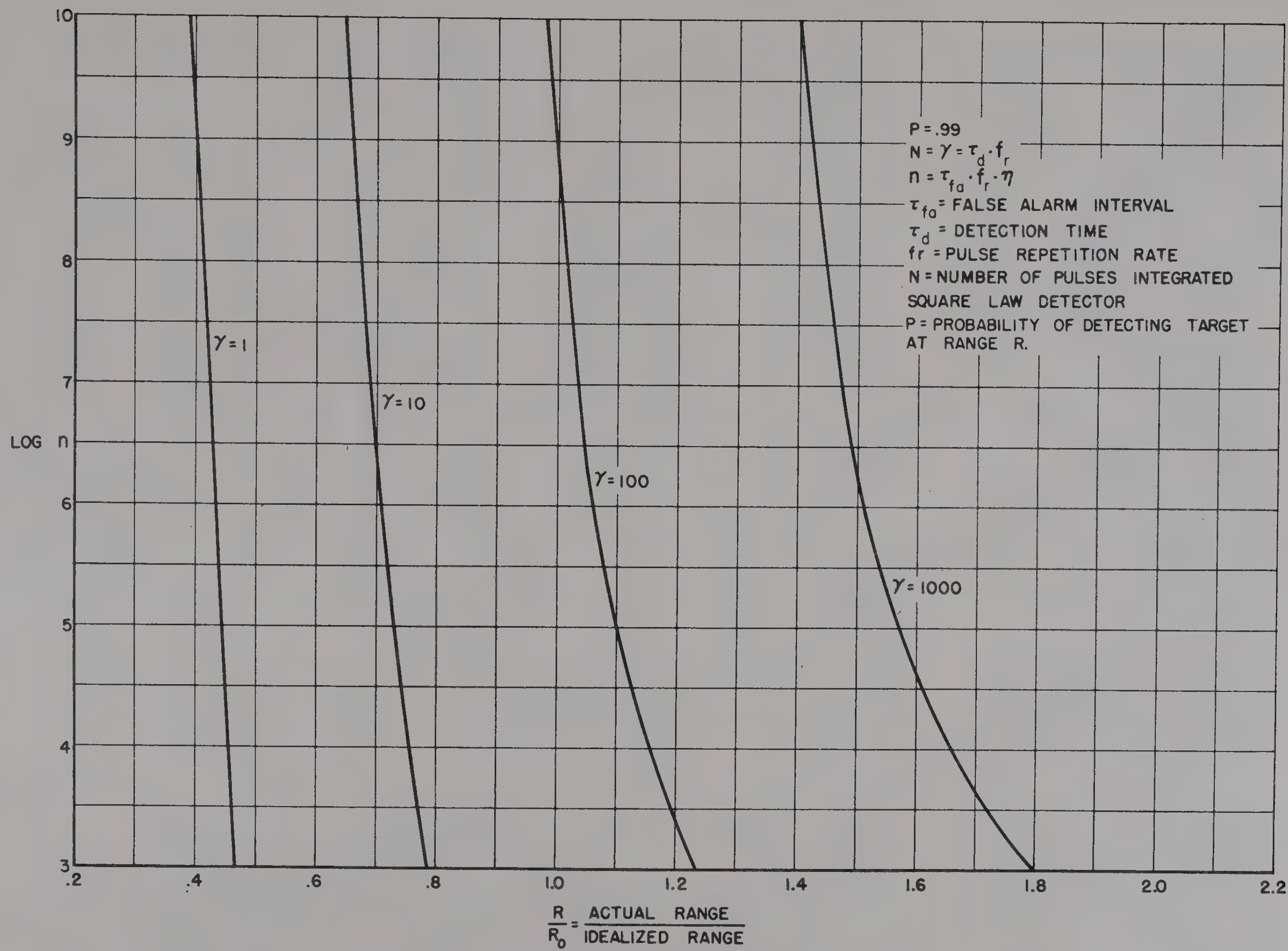

P = .99
N = γ = τd · fr
n = τfa · fr · η
τfa = FALSE ALARM INTERVAL
τd = DETECTION TIME
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
γ = 1
γ = 10
γ = 100
γ = 1000
LOG n
10
9
8
7
6
5
4
3
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
R/R0 = ACTUAL RANGE / IDEALIZED RANGE

FIG. 47

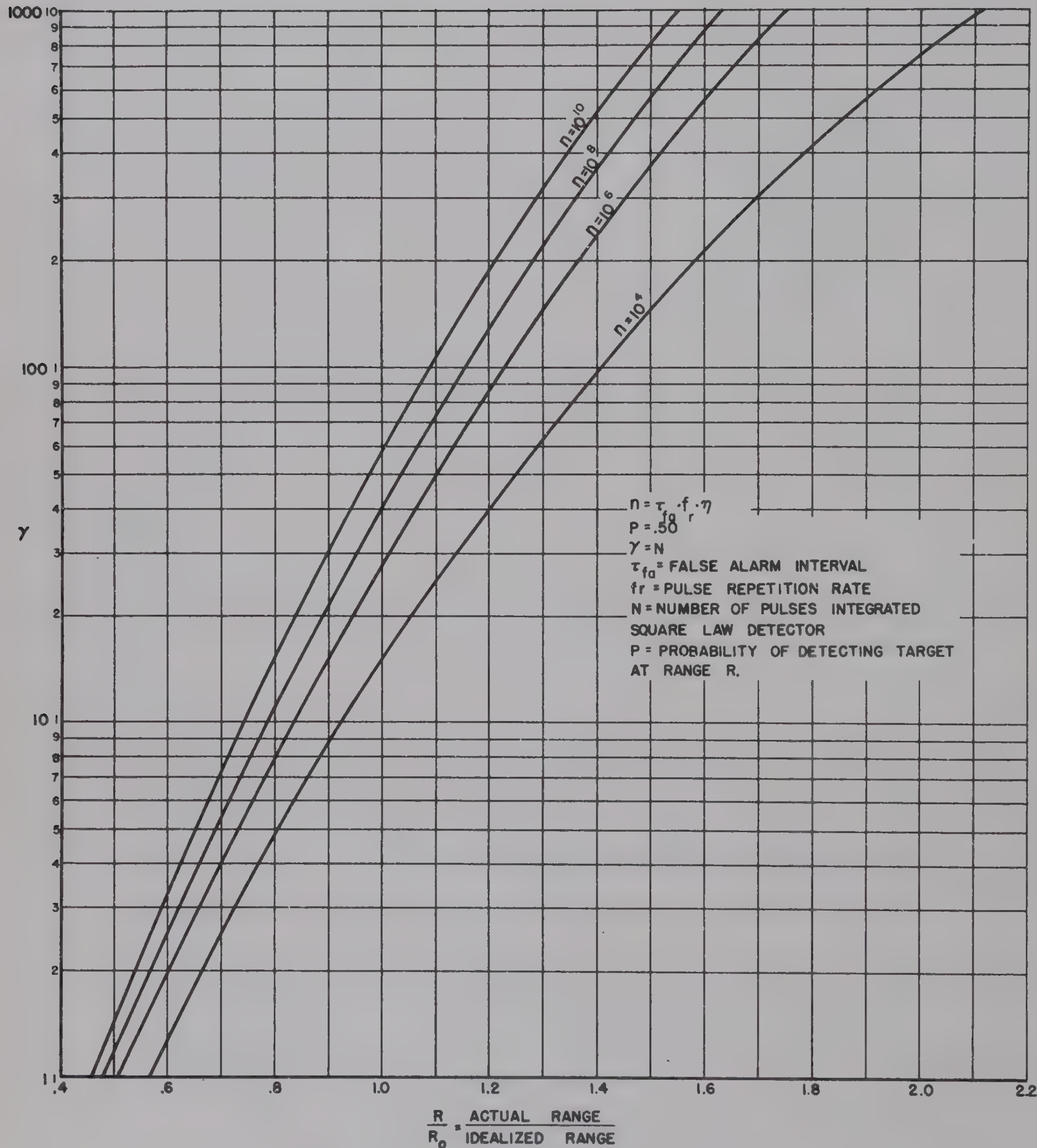

n=10^10
n=10^8
n=10^6
n=10^4
n = τ_fa · f_r · η
P = .50
γ = N
τ_fa = FALSE ALARM INTERVAL
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
γ
1000
100
10
1
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
R/R_0 = ACTUAL RANGE / IDEALIZED RANGE

FIG. 48

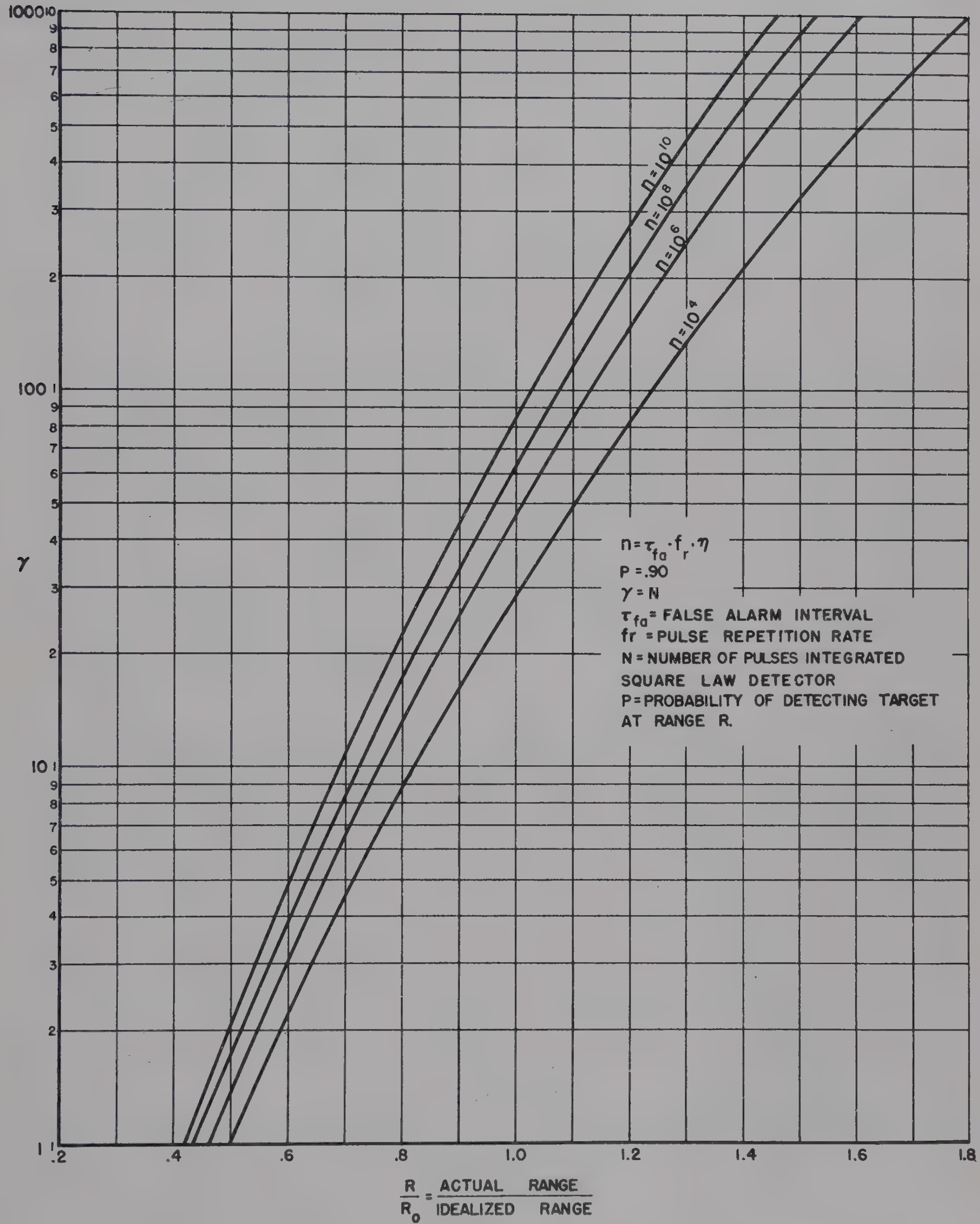

n = 10^10
n = 10^8
n = 10^6
n = 10^4
n = τfa · fr · η
P = .90
γ = N
τfa = FALSE ALARM INTERVAL
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
γ
R/R0 = ACTUAL RANGE / IDEALIZED RANGE

FIG. 49

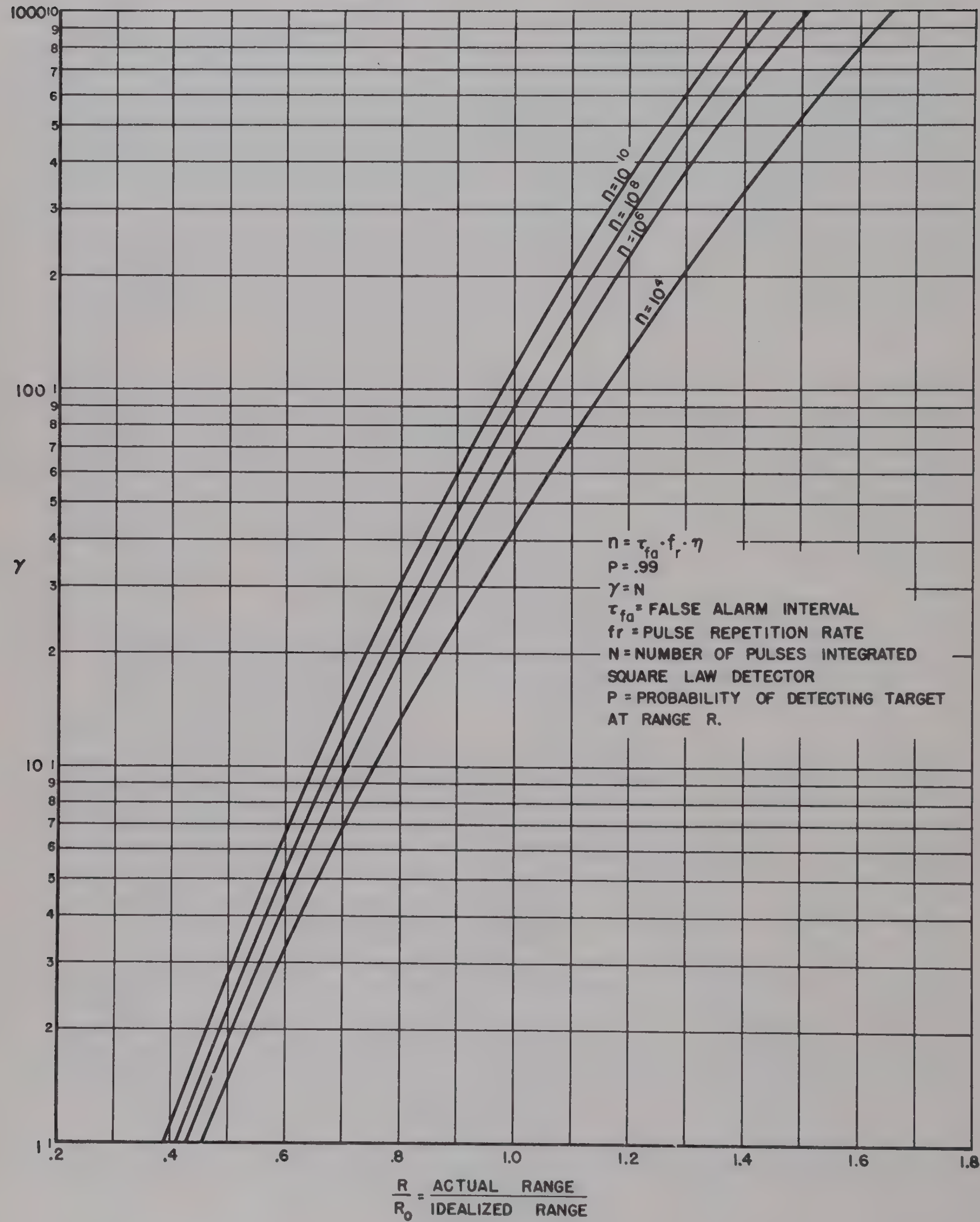
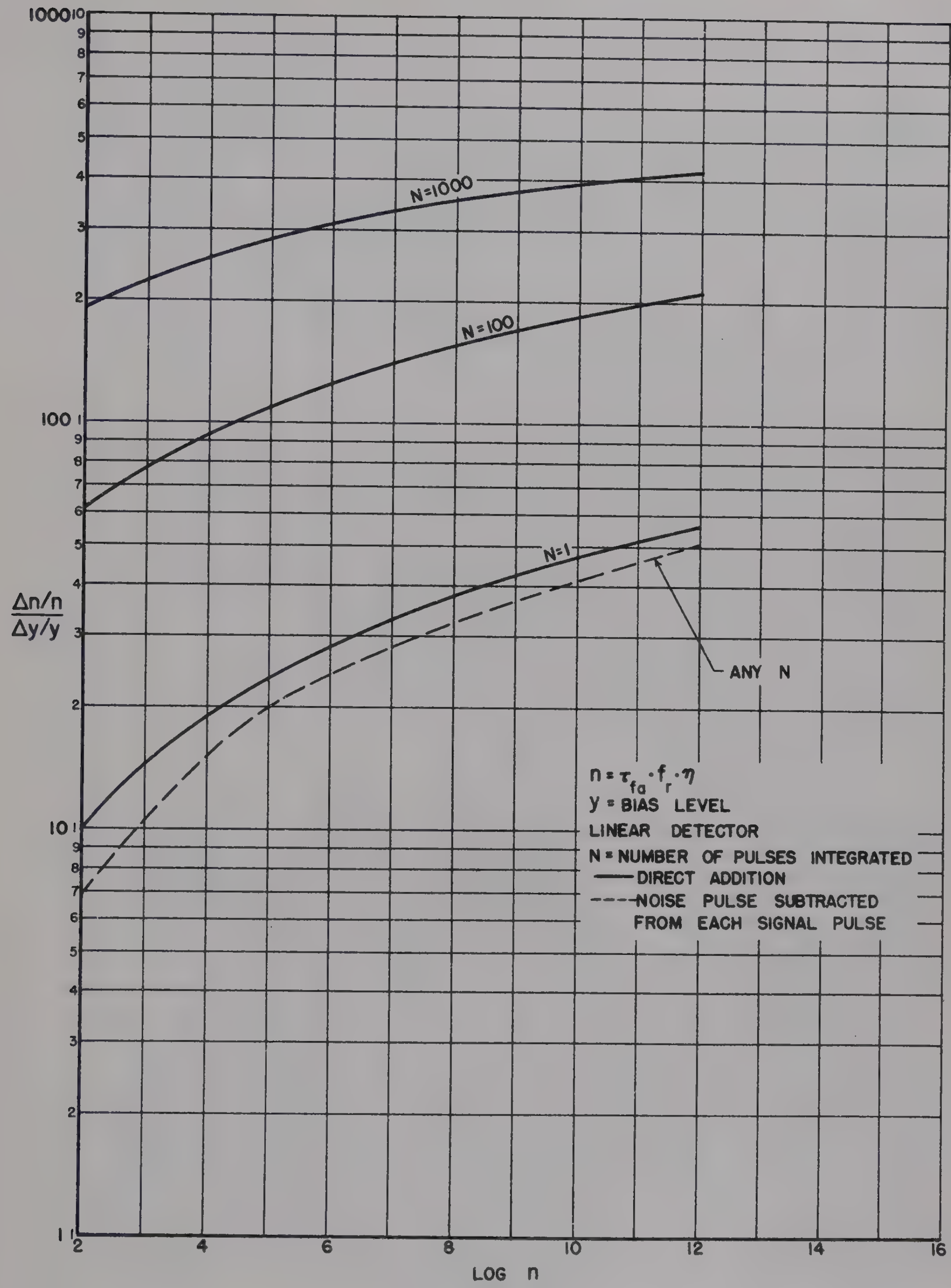
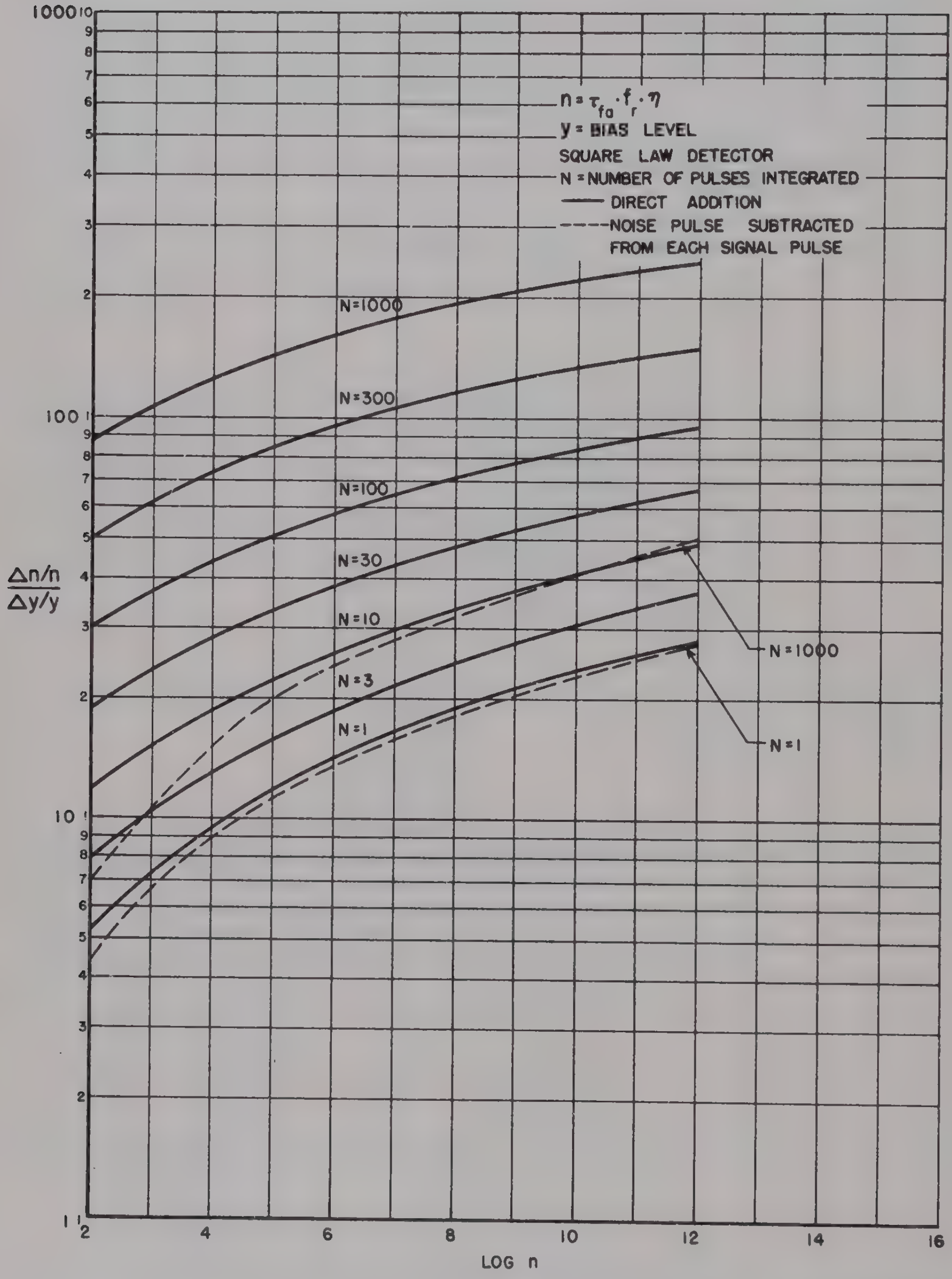

n = 10^10
n = 10^8
n = 10^6
n = 10^4
n = τ_fa · f_r · η
P = .99
γ = N
τ_fa = FALSE ALARM INTERVAL
fr = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
γ
1000
100
10
1
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
R/R_0 = ACTUAL RANGE / IDEALIZED RANGE

FIG. 50



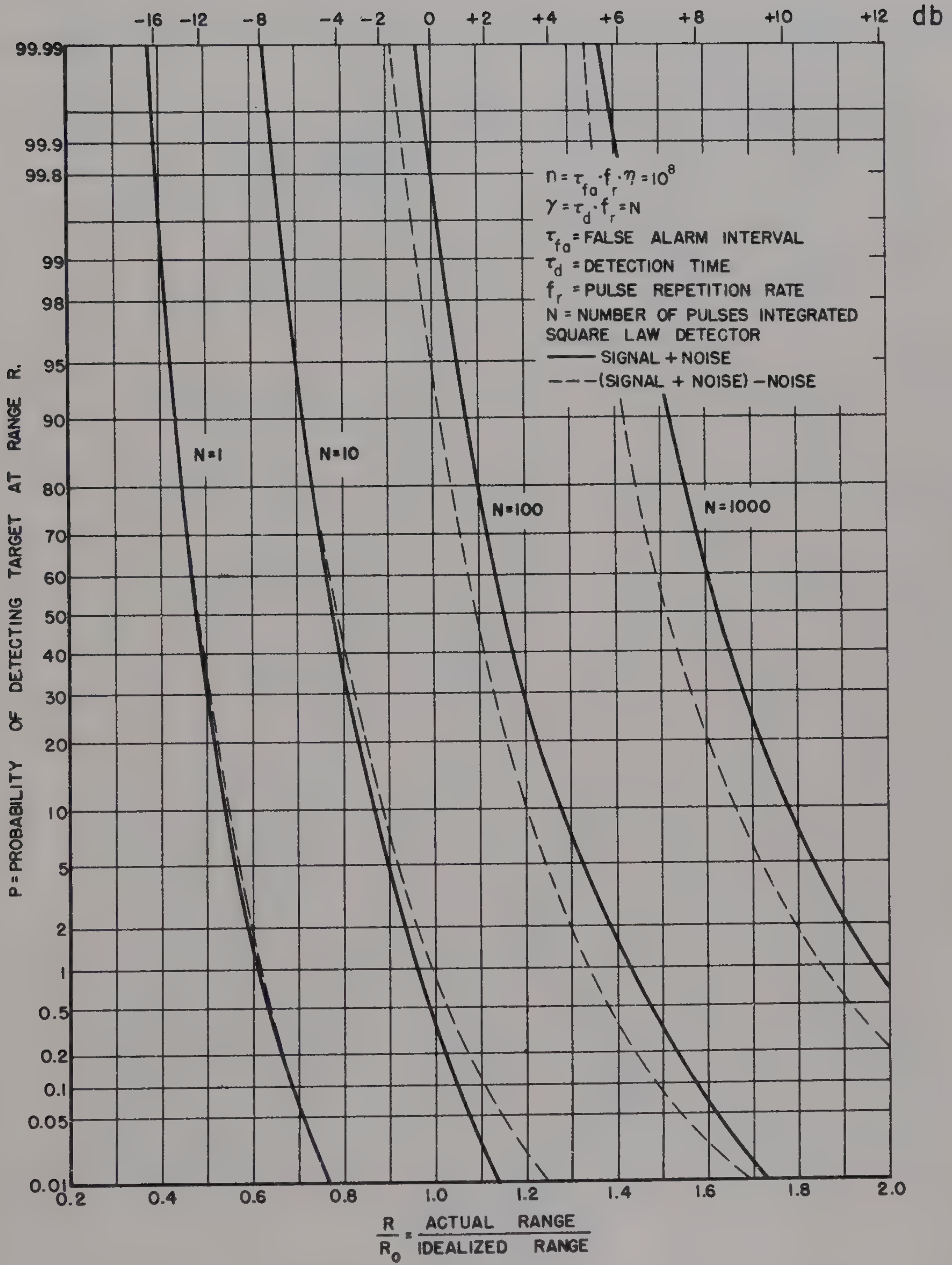
N=1000
N=100
N=1
ANY N
$n = \tau_{fa} \cdot f_r \cdot \eta$
y = BIAS LEVEL
LINEAR DETECTOR
N = NUMBER OF PULSES INTEGRATED
DIRECT ADDITION
NOISE PULSE SUBTRACTED FROM EACH SIGNAL PULSE
$\frac{\Delta n/n}{\Delta y/y}$
LOG n

FIG. 51



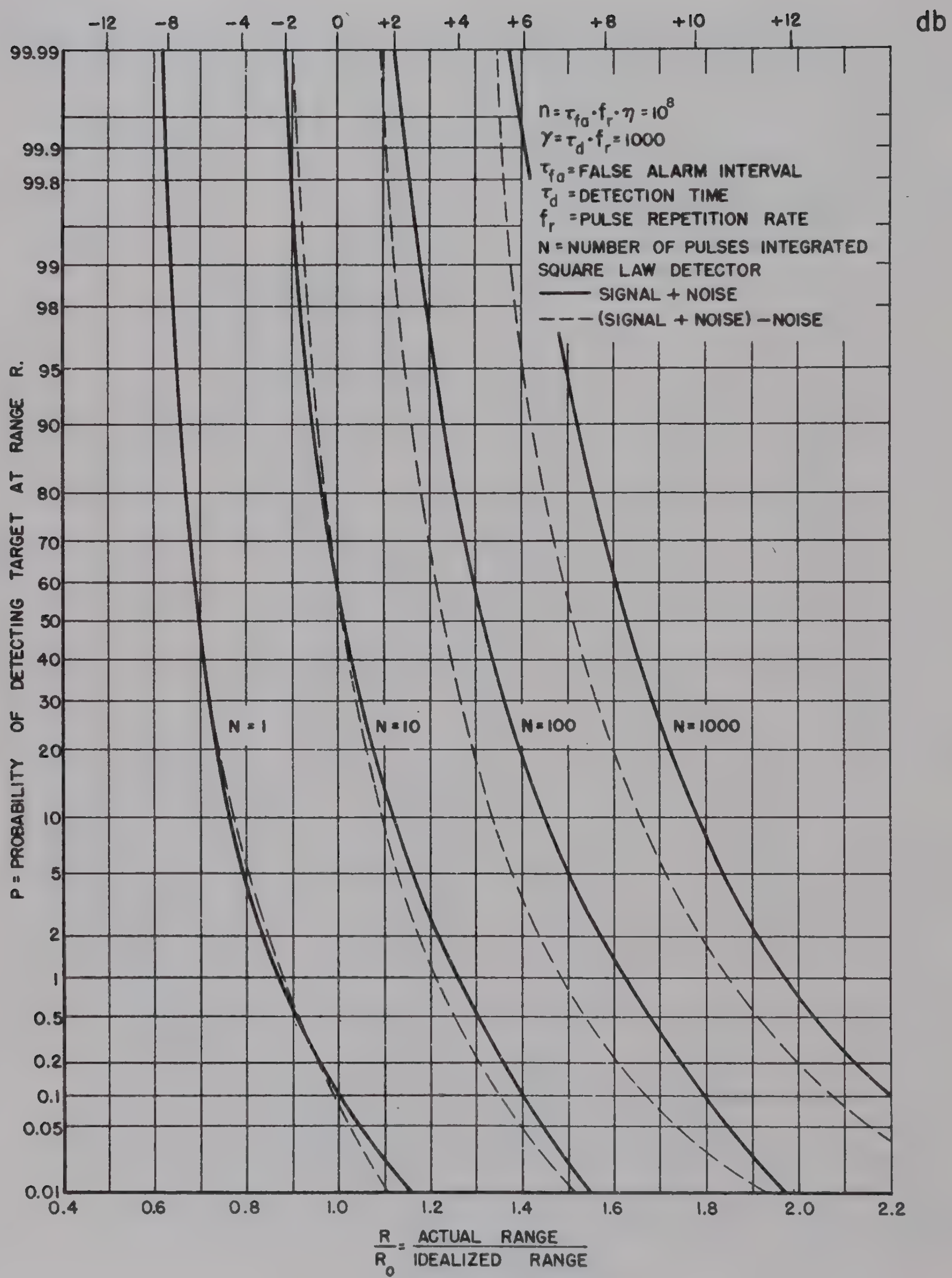
n = τ_fa · f_r · η
y = BIAS LEVEL
SQUARE LAW DETECTOR
N = NUMBER OF PULSES INTEGRATED
DIRECT ADDITION
NOISE PULSE SUBTRACTED FROM EACH SIGNAL PULSE
N=1000
N=300
N=100
N=30
N=10
N=3
N=1
N=1000
N=1
Δn/n / Δy/y
LOG n

FIG. 52



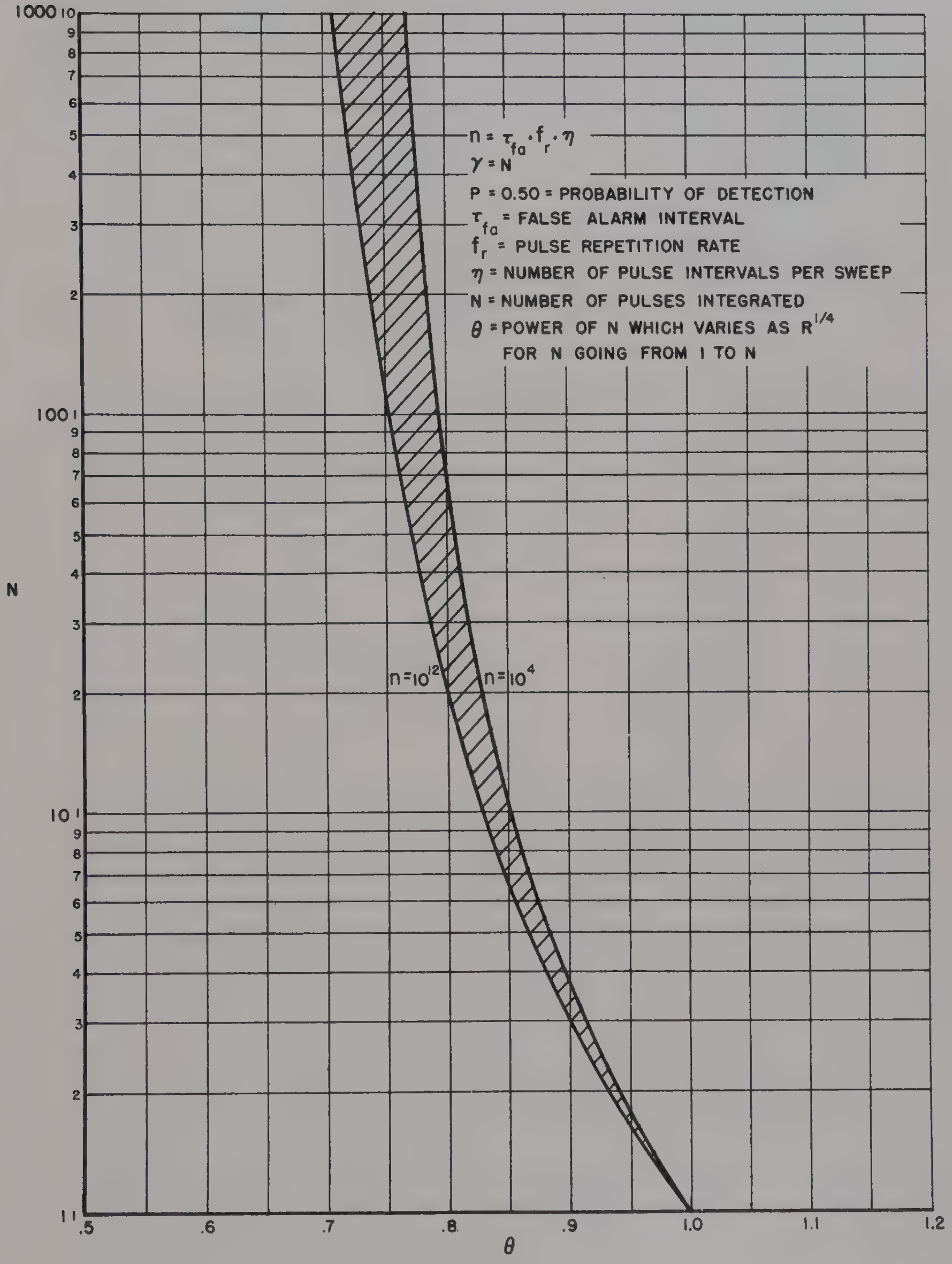
db
-16
-12
-8
-4
-2
0
+2
+4
+6
+8
+10
+12
$n = \tau_{fa} \cdot f_r \cdot \eta = 10^8$
$\gamma = \tau_d \cdot f_r = N$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
f_r = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
SIGNAL + NOISE
(SIGNAL + NOISE) − NOISE
N=1
N=10
N=100
N=1000
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
99.99
99.9
99.8
99
98
95
90
80
70
60
50
40
30
20
10
5
2
1
0.5
0.2
0.1
0.05
0.01
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
$\frac{R}{R_o} = \frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$

FIG. 53



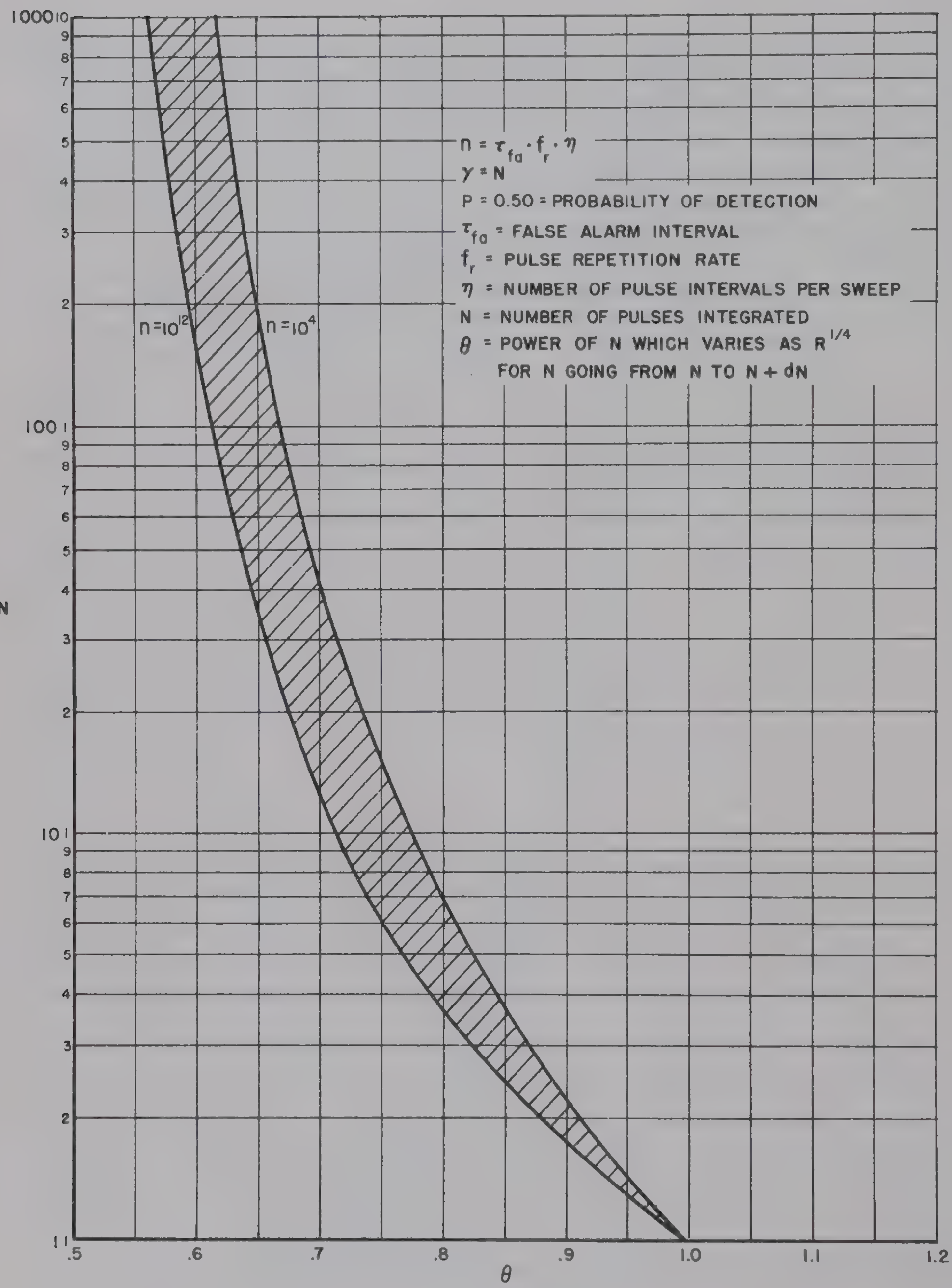
db
-12
-8
-4
-2
0
+2
+4
+6
+8
+10
+12
$n = \tau_{fa} \cdot f_r \cdot \eta = 10^8$
$\gamma = \tau_d \cdot f_r = 1000$
τ_{fa} = FALSE ALARM INTERVAL
τ_d = DETECTION TIME
f_r = PULSE REPETITION RATE
N = NUMBER OF PULSES INTEGRATED
SQUARE LAW DETECTOR
SIGNAL + NOISE
(SIGNAL + NOISE) — NOISE
N = 1
N = 10
N = 100
N = 1000
P = PROBABILITY OF DETECTING TARGET AT RANGE R.
99.99
99.9
99.8
99
98
95
90
80
70
60
50
40
30
20
10
5
2
1
0.5
0.2
0.1
0.05
0.01
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
$\frac{R}{R_0} = \frac{\text{ACTUAL RANGE}}{\text{IDEALIZED RANGE}}$

FIG. 54



n = τfa · fr · η
γ = N
P = 0.50 = PROBABILITY OF DETECTION
τfa = FALSE ALARM INTERVAL
fr = PULSE REPETITION RATE
η = NUMBER OF PULSE INTERVALS PER SWEEP
N = NUMBER OF PULSES INTEGRATED
θ = POWER OF N WHICH VARIES AS R1/4
FOR N GOING FROM 1 TO N
n=10¹²
n=10⁴
N
θ
1000
100
10
1
.5
.6
.7
.8
.9
1.0
1.1
1.2

FIG. 55



1000
100
10
1
N
.5
.6
.7
.8
.9
1.0
1.1
1.2
θ
n=10^12
n=10^4
n = τ_fa · f_r · η
γ = N
P = 0.50 = PROBABILITY OF DETECTION
τ_fa = FALSE ALARM INTERVAL
f_r = PULSE REPETITION RATE
η = NUMBER OF PULSE INTERVALS PER SWEEP
N = NUMBER OF PULSES INTEGRATED
θ = POWER OF N WHICH VARIES AS R^1/4
FOR N GOING FROM N TO N + dN

FIG. 56

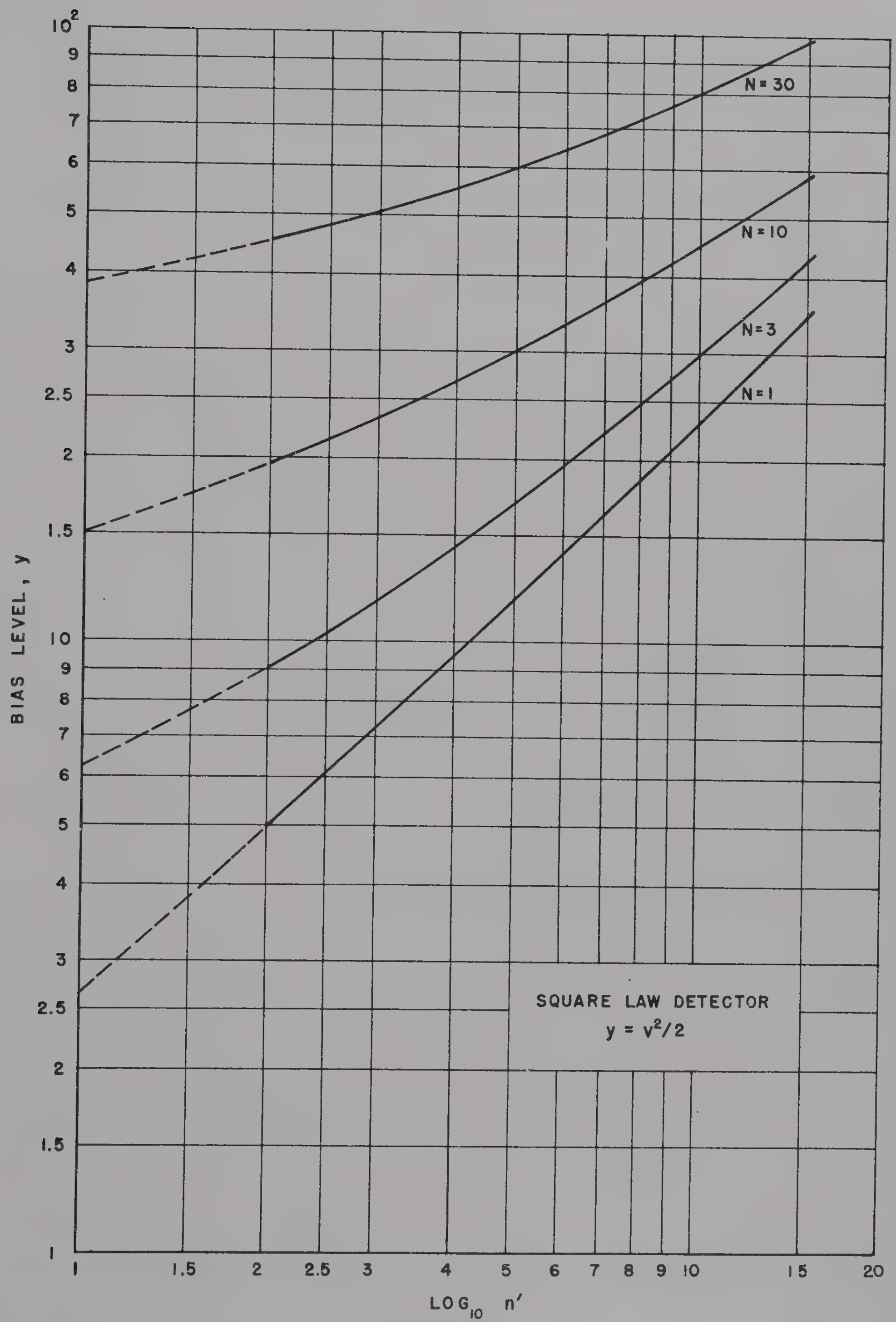


FIG. 57

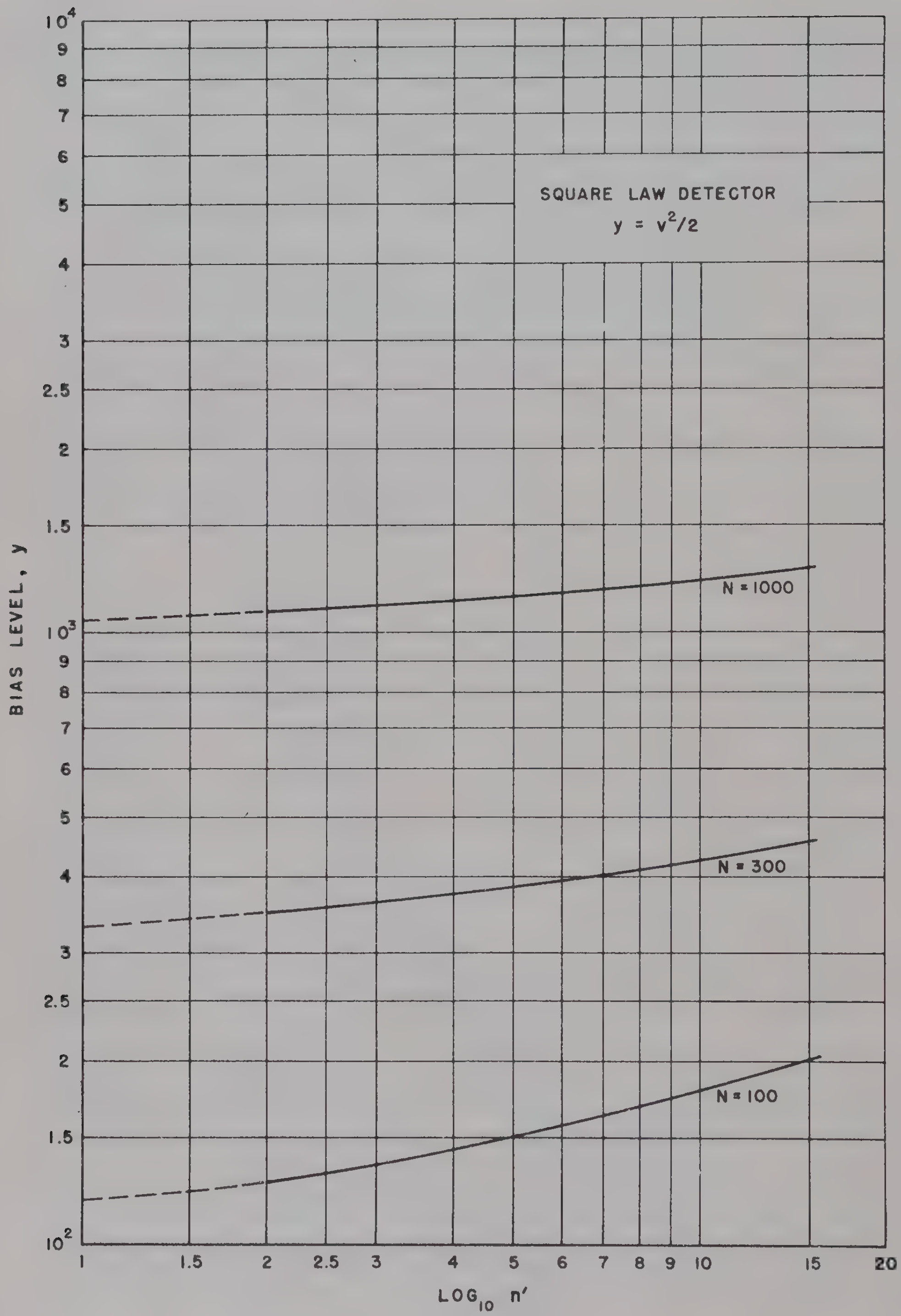

SQUARE LAW DETECTOR
$y = v^2/2$
BIAS LEVEL, y
10^4
9
8
7
6
5
4
3
2.5
2
1.5
10^3
9
8
7
6
5
4
3
2.5
2
1.5
10^2
N = 1000
N = 300
N = 100
1
1.5
2
2.5
3
4
5
6
7
8
9
10
15
20
$LOG_{10}\ n'$

FIG. 57a

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Reference 23 has been published as
Research Memorandum RM-753

Mathematical Appendix

(RM-753, July 1, 1948)

CONTENTS

REFERENCE FOOTNOTES

LIST OF FIGURES

ERRATA

In Figs. 13 thru 24, all of the ordinates appear as percentages but are labeled as probabilities. Therefore, in order to make the two scales conform, the decimal place should be moved two units to the left on all the ordinates.

SUMMARY

In a previous report[28] a statistical theory of radar detection was presented in outline form. The mathematical details were omitted, in order that the main ideas and results might be made available as soon as possible.

This report contains the mathematics that led to the results presented in Ref.28.

In addition, several subjects are briefly discussed that were not covered in Ref.28. These are collapsing loss, antenna beam shape loss, the effect of signal injection, limiting loss, and moving target indication.

For references see page 264.

SYMBOLS

a = amplitude of sine wave relative to R.M.S. noise level

α** = one of the independent variables in the function $Q(\alpha,\beta)$

α_i = i^{th} central standard moment

β** = one of the independent variables in the function $Q(\alpha,\beta)$

B = half-power antenna beamwidth

c_i = coefficient in the Gram-Charlier series

C* = characteristic function

δ_{ij} = delta function

e = base of natural logarithms

f = frequency

${}_1F_1$ = confluent hypergeometric function

F* = Campbell and Foster notation for characteristic function

G* = Campbell and Foster notation for anticharacteristic function

Γ = the gamma function

Γ_N = probability that the sum of N noise variates will exceed the bias level

H_i = i^{th} Hermite polynomial

i = index, subscript, or $\sqrt{-1}$

I = incomplete gamma function as defined by Pearson[8]

I_n = modified Bessel function of the first kind

J_ν = Bessel function of the first kind

κ_i = i^{th} cumulant

K_i** = standard i^{th} cumulant, or sometimes a modified Bessel function of the second kind

L_i = integration loss

L_c = collapsing loss

L_i^α = generalized Laguerre polynomial

M = number of excess noise variates integrated with N signal plus noise variates

n = false alarm number

n' = n/N

N = number of variates integrated

*This symbol has a different meaning in RA-15061.

**This symbol is used in more than one sense in various places, but other meanings should be obvious.

p^* = $i\omega = \omega\sqrt{-1}$

P = sine wave amplitude

P^{**} = probability

P_0 = probability that noise will exceed the bias level at least once within false alarm time

P_N = probability that the sum of N variates of signal plus noise will exceed the bias level

ϕ^i = i^{th} derivative of the error function

ψ_0 = R.M.S. noise level

$Q(\alpha,\beta)$ = modified Lommel's function

R^{**} = envelope amplitude or radar range in R/R_0

R_0 = idealized radar range

ρ = collapsing ratio, ratio of total number of variates integrated to those containing signal

s = cathode ray writing speed

σ^* = standard deviation

T^* = $(y-\bar{y})/\sigma$, semi-independent variable in Gram-Charlier series

T_i^* = incomplete Toronto function

μ_i = i^{th} moment about the mean

U_i = Lommel's function

v^* = normalized envelope amplitude

ν_i = i^{th} moment

$w(f)$ = power spectrum

ω = $2\pi f$

x = power signal-to-noise ratio

y^* = normalized detector output

Y^{**} = integrator output for the sum of N variates

Y_b = bias level

---

*This symbol has a different meaning in RA-15061.

**This symbol is used in more than one sense in various places, but other meanings should be obvious.

A STATISTICAL THEORY OF TARGET DETECTION BY PULSED RADAR: *MATHEMATICAL APPENDIX*

BASIC FORMULAE RELATING TO THERMAL NOISE

Both the thermal noise voltage across a resistor and the noise voltage due to the shot effect in a vacuum tube approach a normal distribution when the number of electrons involved per second in the processes tends toward infinity. In practice, it may usually be assumed that the total noise voltage between any two points due to any combination of thermal, shot, and cosmic noise sources can be represented by the distribution function

$$dP = \frac{1}{\sqrt{2\pi\psi_0}}\, e^{-\frac{V^2}{2\psi_0}}\, dV \tag{1}$$

where ψ_0 is the mean square value of the noise voltage[18]. This distribution is valid provided all elements involved in the composition of the total noise voltage have been linear.

If such noise is now passed through a linear filter of center frequency f_m, having a pass band which is narrow compared to f_m , the output will have an envelope, which has a probability density function

$$dP = \frac{R}{\psi_0}\, e^{-\frac{R^2}{2\psi_0}}\, dR \tag{2}$$

For references see page 264.

where R is the amplitude of the envelope and ψ_0 is the mean square noise voltage, given by

$$\psi_0 = \int_0^\infty w(f)\,df \tag{3}$$

$w(f)$ is the so-called power spectrum of the filter and is simply the square of the absolute value of the amplitude transfer function of the filter.

If the input to a filter consists of a sine wave of frequency f_s, as well as noise, then the probability density function of the output envelope is*

$$dP = \frac{R}{\psi_0}\, e^{-\frac{R^2+P^2}{2\psi_0}}\, I_0\!\left(\frac{RP}{\psi_0}\right) dR, \qquad R > 0$$
$$dP = 0 \qquad\qquad R < 0 \tag{4}$$

where P is the amplitude that the sine wave would have at the output of the filter in the absence of noise, and I_0 is a modified Bessel function of the first kind (see footnote, page 167).

The envelope of the output has a correlation time which is approximately equal to the reciprocal of the bandwidth of the filter. In simple language, it is improbable that the envelope will change by an appreciable percentage in times much less than the correlation time, but it is quite probable that it will change by a goodly percentage in times large compared with the correlation time. It is probably a good approximation to assume that values of the envelope $1/\Delta f$ seconds apart are independent, where Δf is the bandwidth of the filter. By assuming such a discrete process it is possible to materially simplify calculations which would be very tedious if exact integration processes were used, while at the same time sufficient accuracy is obtained for most practical purposes.

A further justification for this assumption in the pulsed case shows in the results. Changing the factor $1/\Delta f$ to $k/\Delta f$ for the correlation time has only the effect of changing the false alarm time by the factor k. The probability of detection turns out to be a very insensitive function of the false alarm time, so that if k is any factor of the order of magnitude of unity, the results are affected to a negligible extent.

---

* It is of some interest to note that the same form of distribution function occurs in other problems. For instance, if ψ_0 represents the mean square velocity of a gas due to ordinary turbulence, and P represents the translational velocity of the whole mass of gas relative to some fixed reference, then the density function of Eq. (4) gives the probability that the total vector velocity at any point in the gas will have a magnitude between R and $R + dR$ [24].

The same density function also represents the probability that a bomb will hit at a distance between R and $R + dR$ from a given point if it is initially aimed at a point whose distance from the given point is P. The mean square aiming error is represented by ψ_0, the distribution being assumed Gaussian [47].

DEFINITION AND EFFECT OF DETECTION

A detector is defined as any device whose instantaneous output is a function of the envelope of the input wave only. Thus

$$y = f\left(\frac{R}{\sqrt{\psi_0}}\right) = F(v) \tag{5}$$

where y is the output of the detector and v is the normalized amplitude of the envelope.

If $P/\sqrt{\psi_0}$ is replaced by a, Eq. (4) may be written

$$\begin{aligned} dP &= ve^{-\frac{v^2+a^2}{2}} I_0(av)\,dv, & v > 0 \\ dP &= 0 & v < 0 \end{aligned} \tag{6}$$

Eq. (5) solved for v is $v = g(y)$. (7)

If v is eliminated from (6) and (7), an equation of the form

$$dP = f(a, y)\,dy \tag{8}$$

is obtained for the probability density for the normalized voltage at the output of the detector which has the characteristics given by Eq. (5). For example, if $y = v^2/2$, then Eq. (8) becomes

$$\begin{aligned} dP &= e^{-y-x} I_0\left(2\sqrt{xy}\right)dy, & y > 0 \\ dP &= 0 & y < 0 \end{aligned} \tag{9}$$

where $a^2/2$ has been replaced by x. The quantity x may be identified with the power signal-to-noise ratio, commonly used in radar literature.

EFFECT OF VIDEO AMPLIFIERS

Since a complete radio receiver usually has one or more stages of video amplification following the detector, it would seem that one would want to calculate the probability density function for signal-plus-noise at the output of the video

amplifier. This can be done theoretically, as has been shown in an excellent paper by Kac[25], but the mathematical labor is great. On the other hand, it has been shown experimentally[36] that the signal threshold is practically unaffected by the video bandwidth until it becomes less than about ½ of the IF bandwidth. Since video bandwidths less than ½ of the IF bandwidth are quite uncommon in practice, it appears best for the sake of simplicity to assume the video bandwidth infinite in all the work which follows.

When the results have been computed, assuming an infinite bandwidth, it will be possible to modify them in an approximate manner so that they become valid for any video bandwidth. This is explained on page 213, under the title "Collapsing Loss".

PROBABILITY OF DETECTION WITH NO INTEGRATION

The calculations necessary to determine the probability of detection when exactly one correlation interval is available are quite simple compared with the case where the output over many correlation intervals is available, and hence the former case is taken up first. In a pulsed system this corresponds to using a single pulse, while in a c-w system it is equivalent to observing the output for a time $t = 1/\Delta f$, where Δf is the over-all effective bandwidth. In either case this amounts to observing the receiver output for one correlation interval. If the output exceeds the bias level, the signal is observed or detected (see pages 9-14 of RA-15061, *A Statistical Theory of Target Detection by Pulsed Radar*[28], hereafter referred to as No.1, for complete definitions of detection and bias level).

It will now be shown that the probability of detecting a given signal x is independent of the detector function, everything else being held constant and only one variate being taken from the density function of Eq.(8). The false alarm time has been defined as the time in which the probability is ½ that the noise alone will not exceed the bias level (Eq.(15), No.1), but it will be best here to keep things general and denote this probability as P_0, rather than as ½. Eq.(15), No.1, then becomes

$$P = 1 - P_0^{N/n} \equiv \Gamma_N \tag{10}$$

where the subscript N denotes the number of variates and Γ is simply an abbreviation. From Eq.(8),

$$\Gamma_1 = \int_{y_b}^{F(\infty)} f(0,y)\, dy \tag{11}$$

where the symbol y_b is now used for the bias number. Then the probability of detection is

$$P_1 = \int_{y_b}^{F(\infty)} f(a,y)\, dy \tag{12}$$

but since $y = F(v)$, or $v = g(y)$, Eq.(11) may be written

$$\Gamma_1 = \int_{g(y_b)}^{\infty} v e^{-\frac{v^2}{2}} dv = e^{-\frac{g^2(y_b)}{2}} \tag{13}$$

and

$$g(y_b) = \sqrt{2 \log_e \frac{1}{\Gamma_1}} \ . \tag{14}$$

Therefore Eq.(12) becomes

$$P_1 = \int_{\sqrt{2 \log_e 1/\Gamma_1}}^{\infty} v e^{-\frac{v^2+a^2}{2}} I_0(av)\, dv \tag{15}$$

which is independent of the detector function.

The integral of Eq.(15) must be evaluated by approximate methods. This function will appear in several places subsequently, and is defined as*

$$Q(\alpha,\beta) = \int_{\beta}^{\infty} v e^{-\frac{v^2+\alpha^2}{2}} I_0(\alpha v)\, dv \ . \tag{16}$$

Footnote on Q Functions

* It does not appear possible to express the Q function in terms of a finite number of known functions. The Q function is similar to Lommel's functions and in fact can be expressed as

$$Q(\alpha,\beta) \equiv 1 - e^{-\frac{\alpha^2+\beta^2}{2}} \left[iU_1(-i\beta^2, i\alpha\beta) - U_2(-i\beta^2, i\alpha\beta) \right]$$

where U_1 and U_2 are Lommel's functions of the first kind. This identity may be proven using the definite integrals given in Watson(1), pages 540 and 541, especially Eq.5 of page 541. By successive integration by parts, the Q function may be expanded in infinite series giving

$$Q(\alpha,\beta) = e^{-\frac{\alpha^2+\beta^2}{2}} \sum_{r=0}^{r=\infty} \left(\frac{\alpha}{\beta}\right)^r I_r(\alpha\beta)$$

or

$$Q(\alpha,\beta) = 1 - e^{-\frac{\alpha^2+\beta^2}{2}} \sum_{r=1}^{r=\infty} \left(\frac{\beta}{\alpha}\right)^r I_r(\alpha\beta) \ .$$

The similarity of the first of these expansions to the series for $U_1(w,z)$ given in Eq.(1), page 537 of Watson, is interesting. A simple expression for $Q(\alpha,\alpha)$ analogous to Eqs.(9) and (10), page 538 of Watson is

$$Q(\alpha,\alpha) = \frac{1}{2}\left[1 + e^{-\alpha^2} I_0(\alpha^2)\right]$$

(*Continued on next page.*)

In terms of this notation, Eq.(15) may be written

$$P_1 = Q\left(a, \sqrt{2 \log_e 1/\Gamma_1}\right) . \tag{17}$$

This is the probability of detection only if $\tau_d = 1/f_r$ where τ_d is the time available for detection. In general the probability of detection is given by Eq.(18), No.1,

$$P_{N,\gamma} = 1 - (1 - P_N)^{\gamma'} \tag{18}$$

which follows from the definition of detection given on page 9, No.1. The double subscript notation $P_{N,\gamma}$ is used here in place of P'. If $N = \gamma$, $P_{N,N}$ is written simply as P_N.

$P_{1,\gamma}$ (R/R_0) can be calculated by means of Eqs.(18), (17), the tables of Q, and the simple relation

$$\frac{R}{R_0} = \frac{1}{x^{1/4}} = \frac{2^{1/4}}{\sqrt{a}} \tag{19}$$

(see Eqs.(10), (11) and (12), No.1).

Footnote on Q Functions (*Cont'd*)

which is useful in special cases. An asymptotic expansion for Q which is of value is given by Rice [18], page 109:

$$Q(\alpha,\beta) = \tfrac{1}{2}[1 - \phi^{-1}(\beta-\alpha)] + \frac{1}{2\alpha\sqrt{2\pi}}\, e^{-\frac{(\beta-\alpha)^2}{2}}\left[1 - \frac{\beta-\alpha}{4\alpha} + \frac{1+(\beta-\alpha)^2}{8\alpha^2} \text{----}\right]$$

where $\phi^{-1}(T)$ is given by the error function of Eq.(100). This expression is most useful in the region where $\alpha\beta \gg 1$ and $\alpha \gg |\beta-\alpha|$.

The Q function is a special case of the incomplete Toronto function described in the footnote on page 182. The relation is

$$Q(\alpha,\beta) = 1 - T_{\frac{\beta}{\sqrt{2}}}\left(1, 0, \frac{\alpha}{\sqrt{2}}\right) .$$

The Q function is graphed in Figs.13 and 14.

A table is available in Ref.47 but the intervals are too large to be of general use. Project RAND is computing an extensive table of the Q function which will be published as a separate report.

A very good approximation for the quantity Γ_N, which appears in Eqs.(10) and (17), may be derived from Eq.(10) by writing

$$P_0^{1/n'} e^{1/n' \log_e P_0} \approx 1 - \frac{1}{n'} \log_e \frac{1}{P_0} \tag{20}$$

which is valid when $n' \gg 1$, a condition nearly always true in practice. Eq.(10) then becomes

$$\Gamma_N \approx \frac{N}{n} \log_e \frac{1}{P_0} \quad . \tag{21}$$

If $P_0 = 1/2$, as is assumed in all the curves in No.1,

$$\Gamma_1 = \frac{0.693}{n} \quad . \tag{22}$$

Eq.(17) may consequently be written

$$P_1 = Q\left[\sqrt{2}\left(\frac{R_0}{R}\right)^2, \sqrt{4.60 \log_{10} n + 0.730}\right] \quad . \tag{23}$$

As an example, let $R/R_0 = 0.595$, and $n = 10^4$. Then $P_1 = Q(4, 4.37)$, which has the value 0.410 from the table given in Ref.47. Note that this is a point on the graph of Fig.1, page 22, No.1.

GENERAL CASE — INTEGRATION OF N INDEPENDENT VARIATES

If the output of the receiver (or filter) can be observed for a length of time much greater than one correlation period, it is of advantage to integrate the output. The simplest concept of an integrator is a device which linearly adds the voltage output of N samples from the detector. The time elapsing between samplings must be at least one correlation period, in order that the samples may be considered to be independent. If the sum of N variates* of signal-plus-noise exceeds the bias level calculated from the probability density function for N variates of noise alone, then the signal is said to be detected.

* Readers with some statistical experience will recognize that here is a case of testing a statistical hypothesis. It is known that the n observations $y_1, y_2,$ ---- y_n come from a universe whose density function $f(y, a)$ depends on the unknown parameters a; it is required to decide, on the basis of these observations, which of the two values a_1 or a_2 is a better estimate for a. If a_1 is the true value of a, let p_1 be the probability of making the mistake of choosing a_2 as the correct value; similarly, if a_2 is the true value, let p_2 be the probability of choosing a_1. Suppose $p_1 = .05$. Then a statistical decision method can be devised for which $p_1 = .05$ and for which p_2 will be less than for any other method with the same p_1. See, for example, Kendall, vol.2, pp.272-275[6].

The integrator may take the sum of the squares of the N variates, or, in general, the sum of N variates where each variate has been processed by some general function. As long as the same *weight* is applied to each variate, the integrator will be called linear. The function which the integrator applies to each variate will be called the *law* of the integrator. *Any* nonlinear integrator will be inferior in operation to a linear integrator with the same law and would ordinarily never be used intentionally in practice. Cathode ray tubes are nonlinear, however, and thus fall short of other types of linear integrations.

The *law* of the integrator acts in exactly the same way as the law of the detector. Thus, if the detector output is $y = (F(v)$ as given by Eq. (5), the integrator output is

$$Y = \sum_{1}^{N} \phi(y) = \sum_{1}^{N} \phi[F(v)]. \tag{24}$$

It is obvious, as far as the theoretical problem is concerned, that the only function of importance is

$$\psi(v) = \phi[F(v)] . \tag{25}$$

There will be an infinite number of combinations of ϕ and F functions which will produce the same function ψ and hence the same theoretical results. In all the work that follows, the output of the combination of integrator and detector for one independent variate will be called $y = \psi(v)$, and the sum of N variates will be

$$Y = \sum_{1}^{N} y . \tag{26}$$

The symbolic solution for the case of N variates corresponding to Eq. (15) for one variate is not too difficult to obtain. The starting point is Eq. (8) for the probability density function for one variate. The characteristic function for this distribution is

$$C_1 = \int_{-\infty}^{\infty} f(a, y) e^{i\omega y} \, dy . \tag{27}$$

The characteristic function for the probability density function for the sum of N independent variates is then simply

$$C_N = (C_1)^N \tag{28}$$

and

$$dP_N = dY \int_{-\infty}^{\infty} C_N(a,\omega)\, e^{-i\omega y} \frac{d\omega}{2\pi} \tag{29}$$

or

$$dP_N = G(a,N,Y)\, dY \; . \tag{30}$$

Corresponding to Eq.(11) is

$$\Gamma_N = \int_{Y_b}^{\infty} G(0,N,Y)\, dY \tag{31}$$

and to Eq.(12),

$$P_N = \int_{Y_b}^{\infty} G(a,N,Y)\, dY \; . \tag{32}$$

If Y_b is eliminated from Eqs.(31) and (32), there results a solution for P_N as a function of Γ_N, N, and a, which is the desired result.

It is found in most cases that the integrations required in Eqs.(27) to (32) are not possible in terms of known functions.

THE SQUARE LAW DETECTOR WITH N VARIATES

It seems, by a process of trial and error, that the best possible function for $\psi(v)$ in Eq.(25) to produce integrable functions in Eqs.(27) to (32) is

$$\psi(v) = Av^2 = y \; . \tag{33}$$

Though this represents a square law for the combined detector and integrator law, it is usual to think of it as representing a square law detector coupled with a linear law integrator.

In Eq.(33), the only effect of the constant A is to multiply the bias level Y_b in Eqs.(31) and (32) by A. The value of P_N in Eq.(32) is independent of A. It is convenient to let $A = 1/2$, or $y = v^2/2$. By direct substitution from Eqs.(6) and (27),

$$C_1 = \int_0^\infty e^{-y-x} I_0(2\sqrt{xy}) e^{py}\, dy \tag{34}$$

where $x = a^2/2$ and $p = i\omega$.

This integral may be obtained from pair 655.1 of Campbell and Foster[7]. In all pairs taken from Campbell and Foster it is necessary to replace p by $-p$, since they use e^{-pg} for the first integration. As long as the same notation is used in both directions, the order of signs is immaterial. In order to avoid confusion, the minus sign will be used in the exponent in the first transformation and the plus sign in the second transformation. Thus all of the characteristic functions which appear hereafter are really $C(-p)$ rather than $C(p)$. In this way there is direct agreement with the Campbell and Foster tables as well as with tables of the Laplace transform. Equation (34) becomes

$$C_1 = \frac{1}{p+1} e^{-x} e^{\frac{x}{p+1}} \;. \tag{35}$$

The characteristic function for the sum of N variates is then simply

$$C_N = \frac{e^{-Nx}}{(p+1)^N} e^{\frac{Nx}{p+1}} \;. \tag{36}$$

By means of pair 650.0, Campbell and Foster, the probability density function is

$$\begin{aligned} dP_N &= \left(\frac{Y}{Nx}\right)^{\frac{N-1}{2}} e^{-Y-Nx} I_{N-1}(2\sqrt{NxY})\, dY \qquad && Y > 0 \\ &= 0 && Y < 0 \;. \end{aligned} \tag{37}$$

Graphs of this function are shown in Figs.1-7. The density function for noise alone ($x = 0$) is found most easily from pair 431, Campbell and Foster, to be

$$dP_N = \frac{Y^{N-1} e^{-Y}}{(N-1)!} dY \;. \tag{38}$$

BIAS LEVEL FOR SQUARE LAW CASE

The bias level, Y_b, is by Eq.(31)

$$\Gamma_N = \int_{Y_b}^{\infty} \frac{Y^{N-1} e^{-Y}}{(N-1)!} \, dY \quad . \tag{39}$$

The incomplete gamma function, as defined by Pearson[8], is

$$\mathrm{I}(u,p) = \int_0^{u\sqrt{p+1}} \frac{e^{-v} v^p \, dv}{p!} \quad . \tag{40}$$

In terms of this function, Eq.(39) becomes

$$\Gamma_N = 1 - \mathrm{I}\left(\frac{Y_b}{\sqrt{N}}, N-1\right) \quad . \tag{41}$$

The tables of $\mathrm{I}(\mu,p)$ extend to $p = 50$, and values of the function are given to seven places. Thus, for $N < 50$, and $n' < 10^6$, the bias level Y_b may be obtained directly from Pearson's tables. Other methods must be evolved for $N > 50$ or $n' > 10^6$. The normal approximation to Eq.(39) is unsatisfactory for N less than several thousand because of the fact that the integral is over a region which is far out on the tail of the curve. This can be seen from the Gram-Charlier series which will be taken up presently.

The integral of Eq.(39) may be evaluated directly by successive integration by parts to give

$$\Gamma_N = \frac{Y_b^{N-1} e^{-Y_b}}{(N-1)!} = \left[1 + \frac{N-1}{Y_b} + \frac{(N-1)(N-2)}{Y_b^2} \;---- \right] . \tag{42}$$

In the regions of interest $Y_b > N >> 1$. The series in the brackets may be approximated by

$$1 + \frac{N-1}{Y_b} + ----- \quad \approx \quad \frac{1}{1-\frac{N-1}{Y_b}} \tag{43}$$

in this region so that Eq.(42) becomes

$$\Gamma_N \approx \frac{Y_b^{N-1} e^{-Y_b}}{(N-1)!\left(1-\frac{N-1}{Y_b}\right)} = \frac{NY_b^N e^{-Y_b}}{N!(Y_b-N+1)} \quad . \tag{44}$$

By the use of Sterling's approximation for $N!$, Eq.(44) reduces to

$$\Gamma_N = \sqrt{\frac{N}{2\pi}} \frac{\exp\left[-Y_b+N\left(1+\log_e \frac{Y_b}{N}\right)\right]}{(Y_b-N+1)} \quad . \tag{45}$$

Though the expression looks more cumbersome in this form, it is actually much simpler to use in calculations than is Eq.(44). Substituting for Γ_N from Eq.(21) gives the expression

$$\log_{10} n = 0.24 + \frac{1}{2}\log_{10} N + \log_{10}(Y_b-N+1) \tag{46}$$

$$+ 0.434\,(Y_b-N) - N\log_{10}\frac{Y_b}{N} \quad .$$

Graphs of this function are shown in Figs.8 and 9. For $N = 1$, the exact expression for Y_b from Eq.(39) is

$$Y_b = 2.3\log_{10} n + 0.37 \tag{47}$$

whereas Eq.(46) for $N = 1$ reduces to

$$Y_b = 2.3\log_{10} n + 0.45 \quad . \tag{48}$$

The difference is seen to be practically negligible.

CUMULATIVE DISTRIBUTION FOR N VARIATES OF SIGNAL PLUS NOISE – SQUARE LAW DETECTOR

Knowing the required bias level for a given false alarm interval, it is now necessary to integrate the density function of Eq.(37) from this value to infinity to give the probability of detection for a signal of strength x, thus

$$P_N = \int_{Y_b}^{\infty} \left(\frac{Y}{Nx}\right)^{\frac{N-1}{2}} e^{-Y-Nx} I_{N-1}(2\sqrt{NxY})\,dY \ . \tag{49}$$

This integral is not soluble directly in terms of well-known functions. The order of the Bessel function* can be reduced in steps of 1 by successive integrations by parts, so that the last remaining integral is of the type given by the Q function

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* The following are some of the useful identities concerning the modified Bessel functions of the first kind:

$$I_n(z) = (-i)^n J_n(iz) = \sum_{r=0}^{r=\infty} \frac{\left(\frac{z}{2}\right)^{n+2r}}{r!\,(n+r)!} = \frac{\left(\frac{z}{2}\right)^n}{N!}\left[1 + \frac{z^2}{2(2n+2)} + \frac{z^4}{2\cdot 4(2n+2)(2n+4)} ----\right]$$

$$I_0(z) = 1 + \frac{z^2}{4} + \frac{z^2}{64} ----$$

Asymptotic expansion:

$$I_n(x) \approx \frac{e^x}{\sqrt{2\pi x}}\left[1 + \frac{1-4n^2}{1!\,(8x)} + \frac{(1-4n^2)(9-4n^2)}{2!\,(8x^2)} ----\right]$$

$$I_n(x) = \frac{1}{\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)}\left(\frac{x}{2}\right)^n \int_0^{\pi} e^{\pm x\cos\psi}(\sin\psi)^{2n}\,d\psi$$

$$zI_n'(z) = nI_n(z) + zI_{n+1}(z) = -nI_n(z) + zI_{n-1}(z)$$

$$I_0'(z) = I_1(z)$$

$$\int z^n I_{n-1}(z)dz = z^n I_n(z)$$

$$\int zI_0(z)dz = zI_1(z)$$

$$\frac{2n}{z} I_n(z) = I_{n-1}(z) - I_{n+1}(z)$$

$$\int e^x I_0(x)dx = xe^x[I_0(x) - I_1(x)]$$

$$\int e^{-x} I_0(x)dx = xe^{-x}[I_0(x) + I_1(x)]$$

$$\int e^x I_1(x)dx = e^x[(1-x)I_0(x) + xI_1(x)]$$

$$\int e^{-x} I_1(x)dx = e^{-x}[(1+x)I_0(x) + xI_1(x)]$$

Relations between the I_n functions and the hypergeometric functions will be found in the footnote on p. 175.

of Eq.(16). An easier way to arrive at the same result is by the use of the characteristic function. To get the cumulative distribution from $-\infty$ to Y of any density function, it is only necessary to find the anticharacteristic function of C/p, where C is the characteristic function of the given density function (see pair 210, Campbell and Foster). Thus from Eq.(36),

$$P_N = 1 - \int_{-\infty}^{\infty} \frac{e^{-Nx}}{p(p+1)^N} e^{\frac{Nx}{p+1}} e^{pY_b} df^* \quad . \tag{50}$$

The term $1/p(p+1)^N$ may be expanded in a series

$$\frac{1}{p(p+1)^N} = \frac{1}{p(p+1)} - \frac{1}{(p+1)^2} - \frac{1}{(p+1)^3} - \cdots - \frac{1}{(p+1)^N} \quad . \tag{51}$$

The mate of the first term of the series, by pairs 210, and 655.1, Campbell and Foster, is

$$e^{-Nx} \int_0^{Y_b} e^{-y} I_0(2\sqrt{Nxy})\,dy \quad . \tag{52}$$

The first two terms of P_N are thus

$$1 - \int_0^{Y_b} e^{-y-Nx} I_0(2\sqrt{Nxy})\,dy \tag{53}$$

$$= \int_{\sqrt{2Y_b}}^{\infty} v e^{-\frac{v^2+2Nx}{2}} I_0(v\sqrt{2Nx})\,dv$$

$$= Q(\sqrt{2Nx}, \sqrt{2Y_b})$$

using the definition of Q from Eq.(16). All the succeeding terms may be obtained by using pair 650.0 Campbell and Foster.

$$\text{Mate of } \frac{e^{\frac{Nx}{p+1}}}{(p+1)^r} \text{ is} \left(\frac{y}{Nx}\right)^{\frac{r-1}{2}} e^{-y} I_{r-1}(2\sqrt{Nxy}) \quad . \tag{54}$$

---

* As in Campbell and Foster, f is here used in place of $\omega/2\pi$ or $p/2\pi i$.

From Eqs.(53) and (54),

$$P_N = Q\left(\sqrt{2Nx}, \sqrt{2Y_b}\right) + e^{-Y_b - Nx} \sum_{r=2}^{r=N} \left(\frac{Y_b}{Nx}\right)^{\frac{r-1}{2}} I_{r-1}\left(2\sqrt{NxY_b}\right) \quad . \tag{55}$$

This form of solution for P_N is practical for numerical calculation only where N is less than about 10.

The characteristic function in Eq.(50) can be expanded in another manner using

$$\frac{1}{p(p+1)^N} = \sum_{r=N+1}^{r=\infty} \frac{1}{(p+1)^r} \quad . \tag{56}$$

This leads to an expression for P_N complementary to that of Eq.(55) of the form

$$P_N = 1 - e^{-Y_b - Nx} \sum_{r=N+1}^{r=\infty} \left(\frac{Y_b}{Nx}\right)^{\frac{r-1}{2}} I_{r-1}\left(2\sqrt{NxY_b}\right) \quad . \tag{57}$$

By equating (55) and (57), one obtains one of the known expansions for Q given in the footnote on page 159. Equations (55) and (57) may also be obtained directly from Eq.(49) by repeated integration by parts. Equation (57) may also be obtained directly from Eq.(55) by means of the identity

$$e^{\frac{x}{2}\left(t + \frac{1}{t}\right)} = \sum_{n=-\infty}^{n=\infty} t^n I_n(x) \tag{58}$$

given in McRobert[8], page 32, and one of the known series for Q.

For the special case $Y_b = Nx$, the function Q of Eq.(55) is simply

$$Q = \frac{1}{2}\left[1 + e^{-2Y_b} I_0(2Y_b)\right] \tag{59}$$

(see footnote, page 159), and Eq.(55) becomes

$$P_N = \frac{1}{2} + e^{-2Y_b}\left[\frac{1}{2}I_0(2Y_b) + I_1(2Y_b) + I_2(2Y_b) \;----\; I_{N-1}(2Y_b)\right] \quad . \tag{60}$$

This formula is useful for checking special points for values of N around 10 or below.

None of the methods developed above are suitable for calculating P_N for large values of N.

In the next section the general method of Gram-Charlier series is developed, which will be useful in a number of succeeding problems concerning distribution functions over large ranges of variation of N.

EXPANSION OF FUNCTIONS IN GRAM-CHARLIER SERIES

The function $\phi(y)$ is defined by

$$\phi(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \quad . \tag{61}$$

The Hermite polynomials may be defined by the relation

$$\phi^i(y) = \frac{(-1)^i}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} H_i(y) \tag{62}$$

where the superscript i stands for the i^{th} derivative with respect to y. The ϕ functions and the Hermite polynomials are biorthogonal, that is

$$\begin{aligned} \int_{-\infty}^{\infty} H_i(y)\phi^j(y) = \delta_{ij} &= 0, i \neq j \\ &= 1, i = j \quad . \end{aligned} \tag{63}$$

Therefore it is possible to expand any reasonable function in a series of the form[5]

$$f(y) = \sum_{i=0}^{i=\infty} a_i \phi^i(y) \quad . \tag{64}$$

The coefficients a_i may be evaluated in a manner analogous to the Fourier series methods by multiplying both sides of Eq. (64) by $H_i(y)$ and integrating from $-\infty$ to ∞. All terms drop out but one, giving

$$a_i = \frac{(-1)^i}{i!} \int_{-\infty}^{\infty} H_i(y) f(y) dy \quad . \tag{65}$$

It is usual to make the substitution $t = y - \bar{y}/\sigma$ before making the expansion, thus causing the second and third terms of the series to vanish. The notation is

$\bar{y} = \nu_1$ = the average value of y, or the first moment

$\sigma^2 = \nu_2 - \nu_1^2$ = the variance

ν_n = the n^{th} moment of the distribution.

$$\nu_n = \int_{-\infty}^{\infty} y^n f(y)dy \tag{66}$$

Equation (64) is replaced by

$$f(y) = g(t) = \sum_{i=0}^{i=\infty} c_i \phi^i(t) \tag{67}$$

and Eq.(65) by

$$c_i = \frac{(-1)^i}{i!}\int_{-\infty}^{\infty} H_i(t)g(t)dt = \frac{(-1)^i}{i!}\int_{-\infty}^{\infty} H_i\left(\frac{y-\bar{y}}{\sigma}\right)f(y)\frac{dy}{\sigma} \quad . \tag{68}$$

It follows at once from Eq.(68) that $c_0 = 1/\sigma$, $c_1 = c_2 = 0$. The moments about the mean, or the central moments, are defined by

$$\mu_i = \int_{-\infty}^{\infty} (y-\bar{y})f(y)dy \tag{69}$$

and the standard moments about the mean by

$$\alpha_i = \frac{\mu_i}{\sigma^i} \quad . \tag{70}$$

The coefficients c_i in Eq.(68) may be easily written in terms of the α's. The first few are

$$c_3 = -\frac{1}{3!}\alpha_3 \tag{71a}$$

$$c_4 = \frac{1}{4!}(\alpha_4-3) \tag{71b}$$

$$c_5 = -\frac{1}{5!}(\alpha_5-10\alpha_3) \tag{71c}$$

$$c_6 = \frac{1}{6!}(\alpha_6-15\alpha_4+30) \tag{71d}$$

$$c_7 = -\frac{1}{7!}(\alpha_7-21\alpha_5+105\alpha_3) \tag{71e}$$

$$c_8 = \frac{1}{8!}(\alpha_8-28\alpha_6+210\alpha_4-315) \tag{71f}$$

$$c_9 = -\frac{1}{9!}(\alpha_9-36\alpha_7+378\alpha_5-1260\alpha_3) \quad . \tag{71g}$$

Formulae for the μ's in terms of the ν's can be obtained directly from Eq.(69), giving

$$\mu_2 = \nu_2 - \nu_1^2 \tag{72a}$$

$$\mu_3 = \nu_3 - 3\nu_2\nu_1 + 2\nu_1^3 \tag{72b}$$

$$\mu_4 = \nu_4 - 4\nu_3\nu_1 + 6\nu_2\nu_1^2 - 3\nu_1^4 \tag{72c}$$

$$\mu_5 = \nu_5 - 5\nu_4\nu_1 + 10\nu_3\nu_1^2 - 10\nu_2\nu_1^3 + 4\nu_1^5 \tag{72d}$$

$$\mu_6 = \nu_6 - 6\nu_5\nu_1 + 15\nu_4\nu_1^2 - 20\nu_3\nu_1^3 + 15\nu_2\nu_1^4 - 5\nu_1^6 \;. \tag{72e}$$

Continuations of this series are obvious.

The process of obtaining the Gram-Charlier expansion is now evident:

1. Find the moments of the distribution.
2. Obtain the central moments from Eq.(72).
3. Obtain the standard central moments from Eq.(70).
4. Obtain the coefficients from Eq.(71).
5. Write the series for $f(y)$ from Eq.(67).

It turns out that the best grouping for the terms of the series of Eq.(67) is different from the natural sequence[5]. Such a regrouped series is termed an "Edgeworth series" and is actually used in this work. The grouping used by Edgeworth is

$$0 \tag{73a}$$

$$0,\ 3 \tag{73b}$$

$$0,\ 3,\ 4,\ 6 \tag{73c}$$

$$0,\ 3,\ 4,\ 6,\ 5,\ 7,\ 9 \;. \tag{73d}$$

This means that if the 0 and 3 terms are used as the first approximation, the addition of terms 4 and 6 gives the next order approximation, and so forth.

MOMENTS OF SIGNAL PLUS NOISE, SQUARE LAW DETECTOR

The moments of a distribution may be obtained by using the characteristic function as a moment generating function*. Thus

$$\nu_i = (-1)^i \left(\frac{d^i C}{dp^i}\right)_{p=0} \tag{74}$$

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* Kendall, p.54[6].

In the case of the distribution function for the sum of N variates of signal plus noise with a square law detector, the characteristic function is given by Eq. (36), and the moments are

$$\nu_i = (-1)^i \left[\frac{d^i}{dp^i} \frac{e^{-Nx} e^{\frac{Nx}{p+1}}}{(p+1)^N} \right]_{p=0} \quad . \tag{75}$$

Though the first few moments may be obtained by direct differentiation, it is better in this case to expand in a McLaurin's series and obtain the coefficient of $p^i/i!$. Thus

$$\frac{e^{\frac{Nx}{p+1}}}{(p+1)^N} = \frac{1}{(p+1)^N} + \frac{Nx}{(p+1)^{N+1}} + \frac{(Nx)^2}{(p+1)^{N+2} \cdot 2!} \quad . \tag{76}$$

The coefficient of $p^i/i!$ is, by direct expansion of each term in Eq. (76),

$$(-1)^i \frac{(N+i-1)!}{(N-1)!} \left[1 + \frac{(N+i)}{N} Nx + \frac{(N+i)(N+i+1)}{N(N+1)} \frac{(Nx)^2}{2!} + \text{----} \right] \tag{77}$$

$$= (-1)^i \frac{(N+i-1)!}{(N-1)!} \, {}_1F_1(N+i, N, Nx) \tag{78}$$

where ${}_1F_1$ is the confluent hypergeometric function.* Thus the moments are

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* The following are some of the useful relations concerning the confluent hypergeometric function:

$${}_1F_1(a,c,z) = \frac{\Gamma(c)}{\Gamma(a)} \sum_{r=0}^{r=\infty} \frac{\Gamma(a+r)}{r!\,\Gamma(c+r)} z^r = 1 + \frac{a}{c}\frac{z}{1!} + \frac{a(a+1)}{c(c+1)}\frac{z^2}{2!} \text{----} = {}_2F_1\left(a,b,c,\frac{z}{b}\right)$$

$$\text{Lim } b \to \infty$$

Asymptotic expansion:

$${}_1F_1(a,c,z) \approx \frac{\Gamma(c) e^z}{\Gamma(a) z^{c-a}} \left[1 + \frac{(1-a)(c-a)}{z\,1!} \text{----} \right]$$

Kummer's first transformation:

$${}_1F_1(a,c,z) = e^z \, {}_1F_1(c-a, c, -z)$$

Kummer's second transformation:

$${}_1F_1(a, 2a, 2z) = e^z \, {}_0F_1\left(a+\frac{1}{2}, \frac{z^2}{4}\right)$$

Recursion relations:

$$a \, {}_1F_1(a+1,c,z) + (a-c) \, {}_1F_1(a-1,c,z) = (z+2a-c) \, {}_1F_1(a,c,z)$$

$$ac \, {}_1F_1(a+1,c,z) + (c-a) z \, {}_1F_1(a,c+1,z) = c(a+z) \, {}_1F_1(a,c,z)$$

(Continued on next page.)

$$\nu_i = \frac{(N+i-1)!}{(N-1)!} e^{-Nx} {}_1F_1(N+i,N,Nx) \tag{79}$$

or

$$\nu_i = \frac{(N+i-1)!}{(N-1)!} {}_1F_1(-i,N,-Nx) \quad . \tag{80}$$

Hypergeometric function (*Cont'd*)

$$a\,{}_1F_1(a+1,c,z) + (1-c)\,{}_1F_1(a,c-1,z) = (a+1-c)\,{}_1F_1(a,c,z)$$

$$c\,{}_1F_1(a-1,c,z) + z\,{}_1F_1(a,c+1,z) = c\,{}_1F_1(a,c,z)$$

$$(a-c)\,{}_1F_1(a-1,c,z) + (c-1)\,{}_1F_1(a,c-1,z) = (z+a-1)\,{}_1F_1(a,c,z)$$

$$(c-a)\,z\,{}_1F_1(a,c+1,z) + c(c-1)\,{}_1F_1(a,c-1,z) = c(z+c-1)\,{}_1F_1(a,c,z)$$

$$\frac{d}{dz}\,{}_1F_1(a,c,z) = \frac{a}{c}\,{}_1F_1(a+1,c+1,z)$$

Relations between hypergeometric functions and other functions:

$${}_1F_1(a,a,z) = e^z$$

$${}_1F_1(a,a+1,-z) = az^{-a}\int_0^z e^{-t}t^{a-1}dt = z^{-a}\Gamma(a+1)\,\mathrm{I}\left(\frac{z}{\sqrt{a}},\ a-1\right)$$

using Pearson's notation for the incomplete gamma function.

$${}_1F_1(1,a+1,z) = e^{-z}z^{a}\Gamma(a+1)\,\mathrm{I}\left(\frac{z}{\sqrt{a}},\ a-1\right)$$

$${}_1F_1\left(\tfrac{1}{2},\tfrac{3}{2},-z^2\right) = \frac{1}{z}\int_0^z e^{-t^2}dt = \frac{\sqrt{\pi}}{2z}\,erf\ z$$

$${}_1F_1(-n,1,z) = L_n(z)$$

(original Laguerre polynomial)

$${}_1F_1(-n,a+1,z) = \frac{n!\,\Gamma(a+1)}{\Gamma(a+1+n)}\,L_n^{a}(z)$$

(generalized Laguerre polynomial)

$${}_1F_1\left(\nu+\tfrac{1}{2},2\nu+1,-z\right) = \frac{2^{2\nu}\Gamma(\nu+1)e^{-\frac{z}{2}}}{(-z)^{\nu}}\,I_\nu\left(-\tfrac{z}{2}\right)$$

$${}_1F_1\left(\tfrac{1}{2},1,-z\right) = e^{-\frac{z}{2}}I_0\left(\tfrac{z}{2}\right)$$

$${}_1F_1\left(\tfrac{1}{2},2,-z\right) = e^{-\frac{z}{2}}\left[I_0\left(\tfrac{z}{2}\right) + I_1\left(\tfrac{z}{2}\right)\right]$$

$${}_1F_1\left(-\tfrac{1}{2},1,-z\right) = e^{-\frac{z}{2}}\left[(1+z)I_0\left(\tfrac{z}{2}\right) + zI_1\left(\tfrac{z}{2}\right)\right]$$

$${}_1F_1\left(\tfrac{3}{2},1,-z\right) = e^{-\frac{z}{2}}\left[(1-z)I_0\left(\tfrac{z}{2}\right) + zI_1\left(\tfrac{z}{2}\right)\right]$$

$${}_1F_1\left(\tfrac{3}{2},2,-z\right) = e^{-\frac{z}{2}}\left[I_0\left(\tfrac{z}{2}\right) - I_1\left(\tfrac{z}{2}\right)\right]$$

Eq.(80) being obtained from Eq.(79) by Kummer's first transformation. The first four moments are

$$\nu_1 = N(1+x) \tag{81a}$$

$$\nu_2 = (Nx)^2 + 2Nx(N+1) + N(N+1) \tag{81b}$$

$$\nu_3 = (Nx)^3 + 3(Nx)^2(N+2) + 3Nx(N+1)(N+2) + N(N+1)(N+2) \tag{81c}$$

$$\begin{aligned}\nu_4 = {} & (Nx)^4 + 4(Nx)^3(N+3) + 6(Nx)^2(N+2)(N+3) \\ & + 4Nx(N+1)(N+2)(N+3) + N(N+1)(N+2)(N+3) \quad .\end{aligned} \tag{81d}$$

The generalized Laguerre polynomial $L_n^{(\alpha)}(z)$ is defined by

$$L_n^{(\alpha)}(z) = \frac{\Gamma(\alpha+1+n)}{n!\Gamma(\alpha+1)} \; {}_1F_1(-n,\alpha+1,z) \quad . \tag{82}$$

Comparing (80) and (82), it is seen that the moments expressed in terms of the Laguerre polynomials are

$$\nu_i = i!L_i^{(N-1)}(-Nx) \quad . \tag{83}$$

Another generating function for these polynomials is available through the relation

$$L_n^{(\alpha)}(z) = \frac{e^z z^{-\alpha}}{n!} \frac{d^n}{dz^n} \left(e^{-z} z^{n+\alpha}\right) \quad . \tag{84}$$

The moments about the mean may be expressed in terms of the moments about the origin by means of Eqs.(72a-e), resulting in:

$$\mu_0 = 1 \tag{85a}$$

$$\mu_1 = 0 \tag{85b}$$

$$\mu_2 = 2Nx + N = N(2x+1) = \sigma^2 \tag{85c}$$

$$\mu_3 = 6Nx + 2N = 2N(3x+1) \tag{85d}$$

$$\mu_4 = 12(Nx)^2 + 12Nx(N+2) + 3N(N+2) \tag{85e}$$

$$\mu_5 = 120(Nx)^2 + 20Nx(5N+6) + 4N(5N+6) \quad . \tag{85f}$$

A generating function for the central moments may be obtained by multiplying the generating function of Eq.(75) by $e^{p\nu_1}$ giving (see pair 207, Campbell and Foster)

$$\mu_i = (-1)^i e^{-Nx} \left[\frac{d^i}{dp^i} \frac{e^{\frac{Nx}{p+1}+p(Nx+N)}}{(p+1)^N} \right]_{p=0} \quad . \tag{86}$$

The moments of Eqs.(85a-f) are most easily obtained by logarithmic differentiation in Eq.(86).

The standard moments about the mean are obtained from Eq.(70), and are

$$\alpha_0 = 1 \tag{87a}$$

$$\alpha_1 = 0 \tag{87b}$$

$$\alpha_2 = 1 \tag{87c}$$

$$\alpha_3 = \frac{2(3x+1)}{N^{1/2}(2x+1)^{\frac{3}{2}}} \tag{87d}$$

$$\alpha_4 = 3 + \frac{6(4x+1)}{N(2x+1)^2} \quad . \tag{87e}$$

There is an approximate method of computing the significant part of α_6 which is based on the fact that c_6 of Eq.(71d) is always nearly equal to $c_3^2/2$ (see page 259, Fry[5]). Thus

$$\alpha_6 \approx 15\,\alpha_4 + 10\,\alpha_3^2 - 30 \tag{88}$$

or

$$\alpha_6 \approx 15 + \frac{10\left(108x^2+78x+13\right)}{N(2x+1)^3} \quad . \tag{87f}$$

For noise alone, the moments are given by

$$\nu_i = \frac{(N+i-1)!}{(N-1)!} \tag{89}$$

and the central moments by

$$\mu_i = \frac{(N+i-1)!}{(N-1)!}\,{}_1F_1(-i,1-i-N,-N) \quad . \tag{90}$$

Equation (90) was obtained from Eq.(86) by putting $x = 0$ and expanding in a series.

THE GRAM-CHARLIER SERIES FOR THE SQUARE LAW CASE

The coefficients of the series may be obtained by use of Eqs.(71a-d) since the standard central moments are now known (Eqs.87a-f). They are:

$$c_0 = 1 \tag{91a}$$

$$c_1 = c_2 = 0 \tag{91b}$$

$$c_3 = -\frac{3x+1}{3N^{1/2}\,(2x+1)^{\frac{3}{2}}} \tag{91c}$$

$$c_4 = \frac{4x+1}{4N(2x+1)^2} \tag{91d}$$

$$c_6 = \frac{(3x+1)^2}{18N(2x+1)^3} \; . \tag{91e}$$

From Eq.(67) the required series is

$$dP = \frac{dy}{\sigma}\left[c_0\,\phi^0(t) + c_3\phi^3(t) + c_4\phi^4(t) + c_6\phi^6(t) ----\right] \tag{92}$$

where

$$t = \frac{y-\nu_1}{\sigma} \; , \tag{93}$$

$$\nu_1 = N(1+x) \tag{94}$$

$$\sigma = \sqrt{N(1+2x)}$$

and the c's are given by Eqs.(91a-e).

Note that the grouping of terms is according to the Edgeworth scheme given in Eq.(73). Note further that as N tends to infinity, all the coefficients go to zero except c_0. Thus

$$dP = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{(y-\nu_1)^2}{2\sigma^2}}\, dy \tag{95}$$

as $N \to \infty$. In terms of N and x

$$dP = \frac{1}{\sqrt{2\pi N(1+2x)}}\; e^{-\frac{[y-N(1+x)]^2}{2N(1+2x)}}\, dy \tag{96}$$

Eq.(95) is precisely a statement of the central limit theorem, and the derivation given is essentially a loose proof of the theorem.

The cumulative distribution is easily obtained from Eq.(92) by means of the simple relation

$$\int \phi^{i}(t)dt = \phi^{i-1}(t) \tag{97}$$

giving

$$P_N = \int_{Y_b}^{\infty} f(y)dy = \frac{1}{2}[1-\phi^{-1}(T)] - c_3\,\phi^2(T) - c_4\,\phi^3(T) - c_6\,\phi^5(T) \text{ ----} \tag{98}$$

where

$$T = \frac{Y_b - \nu_1}{\sigma} \tag{99}$$

and

$$\phi^{-1}(T) = \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} e^{-\frac{a^2}{2}}\, da \quad . \tag{100}$$

The function $\phi^{-1}(T)$ is tabulated in the W.P.A. tables[9]. This differs from the definition given for $\phi^{-1}(y)$ in Fry, page 456 but is used here because of the W.P.A. tables.

The series of Eq.(98) was used to calculate all the curves of Figs.1-50, No. 1, with the exception of the cases where $N = 1$. In most cases the first two terms of the series are sufficient, though in some regions of small P four terms are needed.

SAMPLE CALCULATION

Assume $N = 10$, $n = 10^6$

From Fig. 8, or Eq.41, $Y_b = 30.0$

Let $\frac{R}{R_0} = 1.0$ so that $x = 1.0$.

From Eq. (94), $\nu_1 = N(1+x) = 20.0$

$$\sigma = \sqrt{N(1+2x)} = 5.48$$

From Eq. (99), $T = \frac{Y_b - \nu_1}{\sigma} = \frac{30.0 - 20.0}{5.58} = 1.830$

$$\phi^{-1}(T) = \phi^{-1}(1.828) = 0.9325$$

$$\tfrac{1}{2}\left(1 - \phi^{-1}(T)\right) = 0.0338$$

From Eq. (91c), $c_3 = -\frac{3x+1}{3N^{1/2}(2x+1)^{\frac{3}{2}}} = -0.081$

From Eq. (91d), $c_4 = \frac{4x+1}{4N(2x+1)^2} = 0.0139$

From Eq. (91e), $c_6 = \frac{(3x+1)^2}{18N(2x+1)^3} = 0.0033$

$\phi^2(1.828) = 0.174$ (See p. 218 for references on tables of the derivatives of the error function.)

$\phi^3(1.828) = -0.470$

$\phi^5(1.828) = 0.990$

$c_3\phi^2(T) = -0.0141$

$c_4\phi^3(T) = -0.0007$

$c_6\phi^5(T) = +0.0032$

$$P = 0.0338 + 0.0141 + 0.0007 - 0.0032 = 0.0452$$

This point, $P = 0.045$, $R/R_0 = 1$, may be found on Fig. 20, No. 1.

INTEGRATION LOSS, SQUARE LAW DETECTOR

It is of interest to express the effect of noncoherent integration as a loss with respect to coherent integration[43]. This may be done by defining the integration loss as

$$L_i = 10 \log_{10} \frac{Nx_2}{x_1} \tag{100a}$$

where

N = number of pulses integrated

x_1 = required value of signal-to-noise ratio to produce given probability of detection for $N = 1$.

x_2 = required value of signal-to-noise ratio to produce the same probability of detection for $N = N$.

Thus L_i is a function of P and n. However, it turns out that the dependence on P and n is very small.

In the case of coherent integration, x_2 is always equal to x_1/N, so that $L_i = 0$. With noncoherent integration, x_2 is always greater than x_1/N, so that noncoherent integration is never as efficient as coherent integration. The results of calculations are given in Figs. 10 and 11. One observes that the dependence of L_i on P and n is quite small. Thus by means of the graph of Fig. 12, which gives x as a function of P and n for $N = 1$, and any one of the curves of Fig. 10, it is possible to obtain a fairly accurate value of x for any P, n and N.

GENERAL CURVES OF THE CUMULATIVE DISTRIBUTION FUNCTION

The integral of Eq. (49) is a function found in other applications than the one discussed in this paper. It is desirable to have graphs of this function available in general form rather than the specialized form of Figs. 1-50, No. 1. The integral is a special case of the incomplete Toronto function* described by Heatley (49) and Fisher (17), which is defined as

$$T_B(m,n,r) = 2r^{n-m+1}e^{-r^2}\int_0^B t^{m-n}e^{-t^2}I_n(2rt)\,dt \quad . \tag{100b}$$

Using this notation, Eq. (49) for the cumulative distribution function becomes

$$P_N = 1 - T_{\sqrt{Y_b}}(2N-1, N-1, \sqrt{Nx}) \quad . \tag{100c}$$

* In normal correlation theory, the quantity

$$df = \left(\frac{B}{i\beta}\right)^{\frac{n_1-2}{2}} e^{-\frac{1}{2}(B^2-\beta^2)} J_{\frac{1}{2}(n_1-2)}(i\beta B)\,d\left(\tfrac{1}{2}B^2\right)$$

is given by Fisher (17) as the limiting form, for large samples, of the frequency element of the quantity $B^2 = n_2R^2$ where R^2 is the sample estimate of the multiple correlation coefficient of a random variable y with other variables x_1, x_2 ---- x_{n1}, n_2 is the size of the sample, and $\beta_2 = n_2\rho^2$ where ρ is the population multiple correlation coefficient. The cumulative distribution is

$$f = T_{\frac{B}{\sqrt{2}}}\left(n_1-1, \tfrac{1}{2}n_1-1, \frac{\beta}{\sqrt{2}}\right)$$

and can be obtained from the curves in Figs. 13 to 32.

The function plotted in Figs. 13 to 32 is

$$T_{\sqrt{V}}(2N-1, N-1, \sqrt{q}) \tag{100d}$$

and P_N may be found easily from these curves for any values of Y_b, N and x.

THE LINEAR DETECTOR – N VARIATES

The linear detector is usually more difficult to deal with than is the square law detector. The distribution function for one variate of signal-plus-noise is

$$dP = ve^{-\frac{v^2+a^2}{2}} I_0(av)dv \ . \tag{101}$$

In attempting to find the distribution for the sum of N variates by the method of characteristic functions, the immediate trouble is that the characteristic function of Eq.(101) does not seem to be obtainable in closed form. To give an idea of the difficulty involved, the characteristic function for one variate of noise alone is obtained as follows:

$$C = e^{-\frac{a^2}{2}} \int_0^\infty ve^{-\frac{v^2}{2} - i\omega v} dv \ . \tag{102}$$

This is pair 903.3, Campbell and Foster, and may be evaluated directly by completing the square or by forming a differential equation, giving in either case

$$C = 1 - \sqrt{\frac{\pi}{2}}\, pe^{\frac{p^2}{2}} erfc\, \frac{p}{\sqrt{2}} \tag{103}$$

or in terms of ω

$$C = 1 - \omega e^{-\frac{\omega^2}{2}} \left[\int_0^\omega e^{\frac{x^2}{2}} dx + i\sqrt{\frac{\pi}{2}} \right] \tag{104}$$

To raise this expression to the N^{th} power and then obtain the anticharacteristic function is practically hopeless.

The distribution function for the sum of two variates of noise alone is obtainable by use of the convolution theorem, giving

$$dP_2 = \frac{y}{2} e^{-\frac{y^2}{2}} + \frac{\sqrt{\pi}}{2} e^{-\frac{y^2}{4}} \left(\frac{y^2}{2} - 1\right) erf\, \frac{y}{2}\, dy \tag{105}$$

and the cumulative distribution is also obtainable, giving

$$P_2 = \int_y^\infty f(y)dy = e^{-\frac{y^2}{2}} + \frac{\sqrt{\pi}}{2} y e^{-\frac{y^2}{4}} erf\, \frac{y}{2} \quad . \tag{106}$$

However, when $N > 2$ there seems to be no closed solution corresponding to Eq.(105) or (106). Since these cases are for noise alone, the signal-plus-noise situation must be attacked by other means.

It turns out that if the moments of the distribution for one variate are known, the moments of the distribution for the sum of N variates may be found directly. Formulae are given, for instance, in Cramer, page 345, [10] for the first few central moments, which are

$$\mu_2^N = N\mu_2 \tag{107a}$$

$$\mu_3^N = N\mu_3 \tag{107b}$$

$$\mu_4^N = N\mu_4 + 3N(N-1)\mu_2^2 \tag{107c}$$

$$\mu_6^N \approx 10N^3\mu_3^3 + 15N^3\mu_2^3 + 15N^2\mu_2\mu_4 - 45N^2\mu_2^3 \quad . \tag{107d}$$

The corresponding coefficients in the Gram-Charlier series then become

$$c_3 = -\frac{\alpha_3}{3!N^{1/2}} \tag{108a}$$

$$c_4 = \frac{\alpha^4 - 3}{4!N} \tag{108b}$$

$$c_6 = \frac{10\alpha_3^2}{6!N} \quad . \tag{108c}$$

The α's in Eqs.(108a-c) are the central standard moments for one variate. Note that, in the square law case, if N is put equal to 1 in Eqs.(85c-f) and the resulting μ's used in Eqs.(107a-d), the μ's for N variates are correctly given. If a moment generating function can be found for the case of N variates, then it is immaterial which method is used; but in the case in which such a function is not available, the Eqs.(107) must be used (or some method essentially equivalent).

To handle the linear detector it is now sufficient to find the moments for one variate only. Rice[18], page 107, gives the required expression as

$$\nu_i = 2^{i/2}\Gamma\left(1+\frac{i}{2}\right){}_1F_1\left(-\frac{i}{2},1,-x\right) . \tag{109}$$

Rice also gives the first two moments as

$$\nu_1 = \sqrt{\frac{\pi}{2}}e^{-\frac{x}{2}}\left[(1+x)I_0\left(\frac{x}{2}\right) + xI_1\left(\frac{x}{2}\right)\right] \tag{110a}$$

$$\nu_2 = 2(1+x) \tag{110b}$$

To calculate ν_3 one needs to know the function, ${}_1F_1(-3/2,1,-x)$. This may be obtained by use of the recursion relation

$$a\,{}_1F_1(a+1,c,z) + (a-c)\,{}_1F_1(a-1,c,z) = (2a+z-c)\,{}_1F_1(a,c,z) \tag{111}$$

by putting $a = -\frac{1}{2}$, $c = 1$, $z = -x$. The result is

$$\nu_3 = 2\nu_1(2+x) - \sqrt{\frac{\pi}{2}}e^{-\frac{x}{2}}I_0\left(\frac{x}{2}\right) \tag{110c}$$

also

$$\nu_4 = 4(2+4x+x^2) . \tag{110d}$$

The corresponding central moments are

$$\mu_2 = \sigma^2 = 2(1+x) - \nu_1^2 \tag{112a}$$

$$\mu_3 = 2\nu_1^3 - 2\nu_1(1+2x) - \sqrt{\frac{\pi}{2}}e^{-\frac{x}{2}} I_0\left(\frac{x}{2}\right) \tag{112b}$$

$$\mu_4 = 4(2+4x+x^2) - 3\nu_1^4 - 4\nu_1\left[(1-x)\nu_1 - \sqrt{\frac{\pi}{2}}e^{-x} I_0\left(\frac{x}{2}\right)\right] \quad . \tag{112c}$$

The standard central moments, and then the c's of Eqs.(108a-c), are directly obtainable from these formulae, though the process is somewhat tedious due to the cumbersome form of Eqs.(112a-c). The functions ν_1 to ν_4 are shown graphically as a function of x in Fig.34.

To obtain the bias level Y_b for the linear detector for $N > 2$, one can use the G.C. series for noise alone. Setting $x = 0$ and $\nu_1 = \sqrt{\pi/2}$ in Eqs.(112a-c) gives

$$\mu_2 = \sigma^2 = 2 - \frac{\pi}{2} = 0.429 \tag{113a}$$

$$\mu_3 = \sqrt{\frac{\pi}{2}}(\pi - 3) = 0.1772 \tag{113b}$$

$$\mu_4 = 8 - \frac{3\pi^2}{4} = 0.598 \tag{113c}$$

and

$$\alpha_3 = \frac{\mu_3}{\sigma^3} = 0.632 \tag{114a}$$

$$c_3 = -\frac{0.1053}{N^{1/2}} \tag{114b}$$

and

$$\alpha_4 = \frac{\mu_4}{\sigma^4} = 3.26 \tag{115a}$$

$$c_4 = \frac{0.0108}{N} \tag{115b}$$

$$c_6 = \frac{0.00555}{N} \quad . \tag{115c}$$

The cumulative distribution function is now equated to Γ_N, giving

$$\frac{0.693N}{n} = \frac{1}{2}\left[1-\phi^{-1}(T)\right] + \frac{0.1053}{N^{1/2}}\phi^2(T) - \frac{0.0108}{N}\phi^3(T) - \frac{0.00555}{N}\phi^5(T) \;--- \tag{116}$$

where

$$T = \frac{Y_b - \nu_1 N}{\sigma\sqrt{N}} = \frac{Y_b - N\sqrt{\frac{\pi}{2}}}{\sqrt{N\left(2-\frac{\pi}{2}\right)}} \quad .$$

For any given n and N, Y_b may be found from Eq.(116) by trial and error methods. If an approximate value of T is found by neglecting all but the first term in Eq.(116), a more accurate value obtained by Newton's method is

$$T_2 = T_1 - \frac{f(T_1)}{f'(T_1)} \quad . \tag{117}$$

It is better, however, to plot Eq.(116) giving n as a function of T and N from which is finally obtained the bias level graph of Fig. 35 showing Y_b as a function of n and N for the linear detector.

Since for finding the bias level it is necessary to know the distribution functions only for large values of the argument, it is possible to find an approximate solution valid in this region. Consider a distribution function given by

$$dP = ve^{-\frac{v^2}{2}}dv \tag{117a}$$

for v going from $-\infty$ to $+\infty$. The N^{th} convolution of this function will be nearly the same for large values as if (117a) went only from 0 to ∞, because the large values in the sum of N variates are most probably produced by addition of large

values of every variate, and for large values (in fact for all positive values) the two distribution functions are identical. The characteristic function of Eq. (117a) is given by pair 710.1 of Campbell and Foster to be

$$C = -\sqrt{2\pi}\; p e^{\frac{p^2}{2}} \quad . \tag{117b}$$

For the sum of N variates

$$C_N = (-1)^N (2\pi)^{\frac{N}{2}} p^N e^{\frac{Np^2}{2}} \quad . \tag{117c}$$

The probability density function is obtained from pair 740.2 of Campbell and Foster as

$$dP_N \approx \frac{(2\pi)^{\frac{N}{2}-1}}{N^{\frac{N+1}{2}}} e^{-\frac{y^2}{4N}} D_N\left(\frac{y}{\sqrt{N}}\right)dy \qquad y >> 1 \tag{117d}$$

where D_N is the parabolic cylinder function of order N. In terms of the derivative of the error integral as defined in Eq.(62),

$$dP_N \approx \frac{(2\pi)^{\frac{N}{2}}}{N^{\frac{N+1}{2}}} \phi^N\left(\frac{y}{\sqrt{N}}\right)dy \qquad y >> 1 \quad . \tag{117e}$$

Note that for $N = 2$, Eq.(117e) becomes

$$dP_2 \approx \frac{\sqrt{\pi}}{2} e^{-\frac{y^2}{4}}\left(\frac{y^2}{2}-1\right) \quad . \tag{117f}$$

Referring to Eq.(105), the exact expression for this case, it is seen that Eq.(117f) can be obtained by neglecting the first term and replacing $erf\ y/2$ by 1, both of these approximations being very good if $y >> 1$.

The approximate cumulative distribution is easily obtained from Eq.(117e) by direct integration and gives

$$P_N \approx \left(\frac{2\pi}{N}\right)^{\frac{N}{2}} \phi^{N-1}\left(\frac{Y}{\sqrt{N}}\right) \quad . \tag{117g}$$

The bias level is easily obtainable from this expression by equating it to Γ_N and solving for Y_b by means of the tables of ϕ^i, or by plotting graphs. The method is not very practical for $N > 20$ since suitable tables do not exist.

It is interesting to note that no such approximation as Eq.(117g) is obtainable for the square law case.

Graphs of the probability density functions for signal-plus-noise have been obtained by numerical convolution for some selected cases and are shown in Figs.36 to 41.

RESULTS OF THE LINEAR DETECTOR CALCULATIONS

The difference in results for the linear and square law detectors turns out to be so small that extreme accuracy must be used in the calculations to show the relation in its true form. One such comparison graph was calculated and is shown in Fig.42. Also, in Fig. 43 is shown the difference in db in the two cases at P = 0.50. The two are identical at N = 1, the linear law becomes better by a maximum of 0.11 db at N = 10, the two are again equal at N = 70, and the square law then becomes better and asymptotically exceeds the linear law by 0.19 db as $N \to \infty$ having reached 0.16 db at N = 1000. These results show conclusively that there is little to choose between the linear law and square law as far as theoretical signal threshold is concerned.

EXPANSIONS IN LAGUERRE SERIES

In certain cases, particularly for low values of N, the Gram-Charlier series may not be the best-suited type of expansion for distribution functions which are zero for all negative values of the amplitude. For low values of N, a suitable expansion for such functions is the following:

$$f(y) = \sum_{i=0}^{i=\infty} a_i\, e^{-y} y^{\alpha} L_i^{\alpha}(y) \tag{118}$$

where $L_i^{\alpha}(y)$ is the generalized Laguerre polynomial defined by Eq.(82), or by

$$L_i^{\alpha}(z) = \frac{e^z z^{-\alpha}}{i!} \frac{d^i}{dz^i} \left(e^{-z} z^{i+\alpha}\right) \quad . \tag{119}$$

The orthogonality relation which makes the expansion possible is

$$\frac{i!}{\Gamma(\alpha+i+1)} \int_0^{\infty} e^{-z} z^{\alpha} L_i^{\alpha}(z)\, L_j^{\alpha}(z) = \delta_{ij} \tag{120}$$

(see Copson[4], page 269).

Thus, from Eqs.(118) and (120), the coefficients are determined by

$$a_i = \frac{i!}{\Gamma(\alpha+i+1)} \int_0^\infty L_i^\alpha(y) f(y)\, dy \quad . \tag{121}$$

Note that Eqs.(118), (120) and (121) are analogous to Eqs.(64), (63) and (65), respectively, for the Gram-Charlier expansion.

Let a new variable $t = y/\beta$. Then

$$f(y) = g(t) = \sum_{i=0}^{i=\infty} c_i e^{-t} t^\alpha L_i^\alpha(t) \tag{122}$$

where

$$c_i = \frac{i!}{\Gamma(\alpha+i+1)} \int_0^\infty L_i^\alpha(t) g(t)\, dt = \frac{i!}{\Gamma(\alpha+i+1)} \int_0^\infty L_i^\alpha\left(\frac{y}{\beta}\right) f(y) \frac{dy}{\beta} \quad . \tag{123}$$

The first few Laguerre polynomials are

$$L_0^\alpha(z) = 1 \tag{124a}$$

$$L_1^\alpha(z) = 1 + \alpha - z \tag{124b}$$

$$2L_2^\alpha(z) = (\alpha+1)(\alpha+2) - 2z(\alpha+2) + z^2 \tag{124c}$$

$$6L_3^\alpha(z) = (\alpha+1)(\alpha+2)(\alpha+3) - 3z(\alpha+2)(\alpha+3) + 3z^2(\alpha+3) - z^3 \quad . \tag{124d}$$

Therefore

$$c_1 = \frac{1}{\Gamma(\alpha+2)\cdot\beta}\left[1 + \alpha - \frac{\nu_1}{\beta}\right] \tag{125}$$

$$c_2 = \frac{1}{\Gamma(\alpha+3)\cdot\beta}\left[(\alpha+1)(\alpha+2) - \frac{2\nu_1}{\beta}(\alpha+2) + \frac{\nu_2}{\beta^2}\right] \quad . \tag{126}$$

Since there are two arbitrary constants, α and β, in the expansion of Eq.(122), it is possible to make $c_1 = c_2 = 0$ by a proper choice of α and β. These relations are easily determined by equating Eqs.(125) and (126) to zero and solving simultaneously. The results are

$$\alpha = \frac{\nu_1^2}{\nu_2 - \nu_1^2} - 1 = \frac{\nu_1^2}{\sigma^2} - 1 \tag{127}$$

$$\beta = \frac{\nu_2 - \nu_1^2}{\nu_1} = \frac{\sigma^2}{\nu_1} \tag{128}$$

and

$$c_0 = \frac{1}{\beta\Gamma(\alpha+1)} = \frac{\nu_1}{\sigma^2\Gamma\left(\frac{\nu_1^2}{\sigma^2}\right)} \tag{129}$$

$$c_1 = c_2 = 0 \tag{130}$$

$$c_3 = \frac{1}{\beta\Gamma(\alpha+4)}\left[\frac{\nu_2}{\beta^2}(\alpha+3) - \frac{\nu_3}{\beta^3}\right] . \tag{131}$$

The coefficients past c_0 are so complicated that the whole value of this type of series seems to depend on the fact that the first term alone is often a good approximation. This approximation is

$$dP = \frac{\nu_1}{\sigma^2\,\Gamma\left(\frac{\nu_1^2}{\sigma^2}\right)}\, e^{-\frac{\nu_1 y}{\sigma^2}} \left(\frac{\nu_1 y}{\sigma^2}\right)^{\frac{\nu_1^2}{\sigma^2}-1} dy \tag{132}$$

and the corresponding cumulative distribution function is

$$P = \int_y^\infty f(y)\,dy = 1 - I\left[\frac{y}{\sigma}, \frac{\nu_1^2}{\sigma^2} - 1\right] \tag{133}$$

where I is the incomplete gamma function as defined by Eq.(40).

There is a striking analogy between Eq.(132) and the corresponding normal approximation. In both cases the distribution for the sum of N variates is simply obtained by multiplying both ν_1 and σ^2 by N. As $N \to \infty$, both the normal approximation and Eq.(132) approach the true distribution (and each other). In any particular case, however, the convergence properties of one approximation will be more useful than the other.

In the square law case, for $x = 0$, $\nu_1 = N$ and $\sigma^2 = N$. Substitution of these values in Eq.(132) gives

$$dP = \frac{1}{\Gamma(N)} e^{-y} y^{N-1} dy \quad . \tag{134}$$

Note that this is the same as Eq.(38), the exact expression. Thus in this particular case the first term gives the whole correct result. The third coefficient from Eq.(131) is easily shown to be zero, as all the following coefficients will be.

In the square law case where $x \neq 0$, $\nu_1 = N(1+x)$ and $\sigma^2 = N(1+2x)$. Substitution of these values in Eq.(132) gives

$$dP = \frac{1+x}{(1+2x)\Gamma\left[\frac{N(1+x)^2}{1+2x}\right]} e^{-\left(\frac{1+x}{1+2x}\right)y} \left[\left(\frac{1+x}{1+2x}\right) y\right]^{\frac{N(1+x)^2}{1+2x} - 1} \tag{135}$$

and from Eq.(133),

$$P = 1 - \mathrm{I}\left[\frac{y}{\sqrt{N(1+2x)}}, \frac{N(1+x)^2}{1+2x} - 1\right] \quad . \tag{136}$$

A comparison of the particular case $N = 3$, $x = 1$ is shown in Fig.44. Curves are given for the exact distribution function (Eq.(37)) and the two approximations given by Eqs.(96) and (135).

For the linear case with $x = 0$, $\nu_1 = N\sqrt{\pi/2}$ and $\sigma^2 = N(2-\pi/2)$, the cumulative distribution is, from Eq.(133),

$$P = 1 - \mathrm{I}\left[\frac{y}{\sqrt{N(2-\pi/2)}}, \frac{N}{4/\pi - 1} - 1\right] \quad . \tag{137}$$

OTHER SERIES APPROXIMATIONS

It is theoretically possible to develop still other series approximations for the various distribution functions. For instance, it might be thought advantageous to use a sum of terms of the type $y^{a}e^{-y^2/2}$, particularly in the linear case. While this turns out to be possible, even the first coefficient is so difficult to calculate that the process is impractical.

METHODS OF INTEGRATION INVOLVING SUBTRACTION OF NOISE

Certain practical difficulties arise in maintaining the bias level at the correct value in an electronic detector, particularly if the number of pulses integrated is large. The trouble may arise from fluctuations in amplifier gain, the bias supply, or the noise level itself.

A solution of this problem is to have the gain of the amplifier, or the bias level, or both, controlled by some sort of average value of the noise output. Obviously the time constant of the control device must be neither too long nor too short. One scheme which has been used is to subtract a pulse known to consist of noise only from each possible signal-plus-noise pulse* (see paragraph 3, page 11, No.1). Thus in the absence of a signal, the average value of any number of composite pulses will always be zero, and the required bias level will be comparatively low.

DISTRIBUTION FUNCTIONS FOR COMPOSITE PULSES OF SIGNAL-PLUS-NOISE MINUS NOISE

When a noise pulse is subtracted from each signal-plus-noise pulse, the theoretical distribution functions will be entirely different from previous cases. The square law case is the only one that can be treated in any reasonable fashion. The distribution function for one variate of signal plus noise is given by

$$dP = e^{-x-Y}\, I_0\left(2\sqrt{xY}\right) dY \tag{138}$$

and the characteristic function is

$$C = \frac{e^{-x}}{p+1}\, e^{\frac{x}{p+1}} \;. \tag{139}$$

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* This subtraction can be accomplished by means of a gate which operates at *double* the repetition frequency. On every other gate only a noise pulse of reversed phase goes through the integrator.

Subtracting a positive noise variate is equivalent to adding a negative noise variate. The distribution function for a negative noise variate is

$$dP = e^Y \qquad Y < 0 \tag{140}$$
$$= 0 \qquad Y > 0$$

and

$$C = \frac{1}{1-p} \tag{141}$$

To obtain the characteristic function for the sum of a variate from the distributions of Eqs.(138) and (140) it is only necessary to take the product of the characteristic functions given by Eqs.(139) and (141), giving

$$C = \frac{e^{-x}}{1-p^2} e^{\frac{x}{p+1}} \tag{142}$$

This is the characteristic function for one so-called composite pulse. The characteristic function for the sum of N composite pulses is simply

$$C = \frac{e^{-Nx}}{(1-p^2)^N} e^{\frac{Nx}{p+1}} \tag{143}$$

In the case of noise alone ($x = 0$),

$$C = \frac{1}{(1-p^2)^N} \tag{144}$$

and the anticharacteristic function is, by pair 569 Campbell and Foster,

$$dP_N = \frac{1}{\sqrt{\pi}(N-1)!} \left|\frac{Y}{2}\right|^{N-\frac{1}{2}} K_{N-\frac{1}{2}} |Y| \, dY \tag{145}$$

where $K_{N-½}$ is a modified Bessel function of the second kind and is given by the finite series

$$K_{N-½}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{r=0}^{r=N-1} \frac{(N+r-1)!}{r!(N-r-1)!(2z)^r} \tag{146}$$

The cumulative distribution for the sum of N composite noise variates may be found by use of the series (146) and term by term integration. However, for N greater than 3 or 4 the process rapidly becomes impractical.

Again, it is necessary to find moments and proceed by means of Gram-Charlier series. For noise alone, the moments are easily found from Eq.(144) to be

$$\nu_i = \mu_i = 0, \quad i \text{ odd} \tag{147}$$

$$\nu_i = \mu_i = \frac{\left(N+\frac{i}{2}-1\right)!(i)!}{(N-1)!\left(\frac{i}{2}\right)!}, \quad i \text{ even}$$

in particular,

$$\mu_2 = 2N = \sigma^2 \tag{148}$$

$$\mu_4 = 12N(N+1) \tag{149}$$

$$\mu_6 = 120N(N+1)(N+2) \tag{150}$$

and

$$\alpha_3 = 0 \tag{151}$$

$$\alpha_4 = 3 + \frac{3}{N} \tag{152}$$

$$\alpha_6 = 15 + \frac{45}{N} + \frac{30}{N^2} \,. \tag{153}$$

The only coefficients different from zero in the first six are c_0 and c_4 to the order of $1/N$.

$$c_4 = \frac{1}{8N} \tag{154}$$

Thus

$$dP_N = \frac{dY}{\sqrt{2N}}\left[\phi^0\left(\frac{Y}{\sqrt{2N}}\right) + \frac{1}{8N}\phi^4\left(\frac{Y}{\sqrt{2N}}\right) \text{----}\right] \tag{155}$$

and similar to Eq.(98) is the cumulative distribution

$$P_N = \frac{1}{2}\left[1-\phi^{-1}\left(\frac{Y}{\sqrt{2N}}\right)\right] - \frac{1}{8N}\phi^{3}\left(\frac{Y}{\sqrt{2N}}\right) \cdots \tag{156}$$

The bias number is found by setting this expression equal to Γ_N and plotting Y_b as a function of n and N. Results are given in Fig. 45. In the special case $N = 1$, the cumulative distribution function is simply $e^{-Y/2}$ for $Y > 0$, and the bias number is obtained from this expression rather than from Eq.(156). The anticharacteristic function of the general case, Eq.(143), may be obtained by use of the convolution theorem, pair 202, Campbell and Foster. Let

$$F_1 = \frac{e^{-Nx}\, e^{\frac{Nx}{p+1}}}{(p+1)^N} \tag{157}$$

$$F_2 = \frac{1}{(1-p)^N} \tag{158}$$

Then from Eq.(37),

$$\begin{aligned} G_1 &= \left(\frac{y}{Nx}\right)^{\frac{N-1}{2}} e^{-y-Nx}\, I_{N-1}\left(2\sqrt{Nxy}\right) \qquad && y > 0 \\ &= 0 && y < 0 \end{aligned} \tag{159}$$

and by pair 525.2, Campbell and Foster,

$$\begin{aligned} G_2 &= 0 && y > 0 \\ &= \frac{(-y)^{N-1} e^{y}}{(N-1)!} && y < 0 \end{aligned} \tag{160}$$

Applying the convolution theorem gives

$$dP_N = dY \frac{e^{Y-Nx}}{(N-1)!} \int_Y^\infty \left(\frac{y}{Nx}\right)^{\frac{N-1}{2}} (y-Y)^{N-1} e^{-2y} I_{N-1}\left(2\sqrt{Nxy}\right) dy \quad Y > 0 \tag{161}$$

For $Y < 0$, the lower limit of the integral in Eq.(161) is 0 rather than Y.

PROBABILITY DENSITY FUNCTIONS FOR $Y < 0$

To evaluate the integral in Eq.(161) when the lower limit is zero is straightforward but tedious. First one evaluates the integral

$$f(k) = \int_0^\infty y^k \left(\frac{y}{Nx}\right)^{\frac{N-1}{2}} e^{-2y} I_{N-1}(2\sqrt{Nxy})\,dy \tag{162}$$

by use of characteristic functions in a manner entirely similar to that used in Eqs.(74) to (80). The characteristic function of the function of Eq.(162), with $k = 0$, is

$$C = \frac{e^{\frac{Nx}{p+2}}}{(p+2)^N} \tag{163}$$

and

$$f(k) = \frac{e^{\frac{Nx}{2}}(N+k-1)!}{(N-1)!\,2^{N+k}}\ {}_1F_1\left(-k, N, -\frac{Nx}{2}\right) \quad . \tag{164}$$

Then by expanding $(y-Y)^{N-1}$, one obtains the coefficient of

$$y^k = \frac{(N-1)!}{k!}\ \frac{(-Y)^{N-1-k}}{(N-1-k)!} \tag{165}$$

and from Eqs.(164), (165), and (161),

$$dP_N = \frac{e^{Y-Nx}}{(N-1)!}\sum_{k=0}^{k=N-1} f(k)\,\frac{(N-1)!}{k!}\ \frac{(-Y)^{N-1-k}}{(N-1-k)!} \tag{166}$$

or

$$dP_N = dY\,\frac{e^{Y-\frac{Nx}{2}}}{(N-1)!}\sum_{k=0}^{k=N-1}\frac{(N+k-1)!\ {}_1F_1\left(-k,N,-\frac{Nx}{2}\right)}{(N-k-1)!\,k!\,2^{N+k}}(-Y)^{N-k-1} \qquad Y<0 \quad . \tag{167}$$

In terms of Laguerre polynomials, using Eq.(82),

$$dP_N = dYe^{Y-\frac{Nx}{2}} \sum_{k=0}^{k=N-1} \frac{L_k^{N-1}\left(-\frac{Nx}{2}\right)}{(N-k-1)!\,2^{N+k}} (-Y)^{N-k-1} \qquad Y < 0 \ . \tag{168}$$

The first few polynomials are given in Eqs.(124a-d). Some special cases of Eq.(168) are

$$N = 1 \qquad dP_1 = dY\frac{e^{Y-\frac{x}{2}}}{2} \qquad Y < 0 \tag{169a}$$

$$N = 2 \qquad dP_2 = dY\frac{e^{Y-x}}{4}\left(1+\frac{x}{2}-Y\right) \qquad Y < 0 \tag{169b}$$

$$N = 3 \qquad dP_3 = dY\frac{e^{Y-\frac{3x}{2}}}{16}\left[3+3x+\frac{9x^2}{16}-\left(2+\frac{3x}{2}\right)Y + Y^2\right] \qquad Y < 0 \ . \tag{169c}$$

The cumulative distributions for $Y < 0$ may easily be obtained by integrating (169a-c). Obviously, the expressions in Eqs.(167) and (168) are practically useful only for small values of N.

PROBABILITY DENSITY FUNCTIONS FOR $Y > 0$

To find a general expression for Eq.(161) giving the distribution function when $y > 0$ is a task of tremendous proportions. Consider, for instance, the special case $N = 1$. Equation (161) becomes

$$dP_1 = dYe^{Y-x}\int_Y^{\infty} e^{-2y}\, I_0\left(2\sqrt{xy}\right)dy \qquad Y > 0 \ . \tag{170}$$

By means of the substitution $y = v^2/4$, this becomes

$$dP_1 = dY\frac{e^{Y-\frac{x}{2}}}{2}\int_{2\sqrt{Y}}^{\infty} e^{-\frac{v^2+x}{2}}\, I_0\left(v\sqrt{x}\right)dv \qquad Y > 0 \ . \tag{171}$$

This can be expressed in terms of the Q function defined by Eq.(16).

$$dP_1 = dY\frac{e^{Y-\frac{x}{2}}}{2} Q(\sqrt{x}, 2\sqrt{Y}) \qquad Y > 0 \ . \tag{172}$$

Eq.(169a) was

$$dP_1 = \frac{e^{Y-\frac{x}{2}}}{2} dY \qquad Y < 0 \ . \tag{169a}$$

Thus, for the case $N = 1$, the whole distribution function is described by Eqs.(169a) and (172). A graph of this function for various values of x is shown in Fig.46. Note that if $x = 0$, $Q(0,2\sqrt{Y}) = e^{-2Y}$, and Eq.(172) for $Y > 0$ reduces to

$$dP_1 = \frac{e^{-Y}}{2} dY \qquad Y > 0 \tag{173}$$

and from Eq.(169a)

$$dP_1 = \frac{e^{Y}}{2} dY \qquad Y < 0 \tag{174}$$

when $x = 0$. Thus over the whole range of Y

$$dP_1 = \frac{e^{-|Y|}}{2} dY \tag{175}$$

which checks Eq.(145) when $N = 1$.

For $N = 2$, Eq.(161) becomes

$$dP_2 = dY e^{Y-2x} \int_Y^\infty \left(\frac{y}{2x}\right)^{\frac{1}{2}} (y-Y) e^{-2y} I_1(2\sqrt{2xy})\, dy \qquad Y > 0 \ . \tag{176}$$

This integral may also be evaluated in terms of the Q function. The process requires a large number of integrations by parts and is very time-consuming. The result turns out to be

$$dP_2 = dY\left[\frac{e^{Y-x}}{4}\left(1+\frac{x}{2}-Y\right)Q(\sqrt{2x}, 2\sqrt{Y}) + \frac{e^{-Y-2x}}{4}\left\{YI_0(2\sqrt{2xY}) + (1+x)\sqrt{\frac{Y}{2x}}\, I_1(2\sqrt{2xY})\right\}\right] \qquad Y > 0 \ . \tag{177}$$

This equation is already so complicated as to be nearly useless. Thus it was not thought worth while to seek a general expression of this type for arbitrary N when $Y > 0$.

Note: If $x = 0$ in Eq.(77), it reduces to

$$dP_2 = dY \frac{e^{-Y}}{4} (1+Y) \qquad Y > 0 \tag{178}$$

which may also be obtained from Eq.(169b) by substituting $-Y$ for Y.

CUMULATIVE DISTRIBUTION FUNCTIONS

The effort in Eqs.(161) to (178) has been concerned with obtaining the probability density function for N variates of signal-plus-noise minus noise. To find the cumulative distribution functions exactly is difficult, especially for Y positive.

A case which can be solved, however, is that for $N = 1$. For Y negative the answer is simply obtained from Eq.(169a) and is

$$P_1 = 1 - \frac{e^{Y-\frac{x}{2}}}{2} \qquad Y < 0 \ . \tag{178a}$$

For Y positive, using the result of Eq.(172),

$$P_1 = \frac{e^{-\frac{x}{2}}}{2} \int_Y^\infty e^y Q(\sqrt{x}, 2\sqrt{y})\, dy \qquad Y > 0 \ . \tag{178b}$$

Since the value of P_1 at $Y = 0$ is, from Eq.(178a), $1 - e^{-\frac{x}{2}}/2$, Eq.(178b) may be rewritten as

$$P_1 = 1 - \frac{e^{-\frac{x}{2}}}{2} - \frac{e^{-\frac{x}{2}}}{2} \int_0^Y e^y Q(\sqrt{x}, 2\sqrt{y})\, dy \tag{178c}$$

but from the definition of Q in Eq.(16),

$$Q(\sqrt{x}, 2\sqrt{y}) = \int_{2\sqrt{y}}^\infty v e^{-\frac{v^2+x}{2}} I_0(v\sqrt{x})\, dv = 2e^{-\frac{x}{2}} \int_y^\infty e^{-2z} I_0(2\sqrt{xz})\, dz \ . \tag{178d}$$

Replacing Q by its defining integral in Eq.(178c) gives

$$P_1 = 1 - \frac{e^{-\frac{x}{2}}}{2} - e^{-x}\int_0^Y dy e^y \int_y^\infty e^{-2z} I_0(2\sqrt{xz})\,dz \,. \tag{178e}$$

Integration by parts is now used, letting

$$\left.\begin{aligned} u &= \int_y^\infty e^{-2z}\, I_0(2\sqrt{xz})\,dz \\ dv &= e^y dy \\ du &= -e^{-2y} I_0(2\sqrt{xy})\,dy \\ v &= e^y \end{aligned}\right\} \tag{178f}$$

$$|uv|_0^Y = e^Y \int_Y^\infty e^{-2z} I_0(2\sqrt{xy})\,dz - \frac{e^{\frac{x}{2}}}{2} \tag{178g}$$

$$= \frac{e^{Y+\frac{x}{2}}}{2} Q(\sqrt{x}, 2\sqrt{Y}) - \frac{e^{\frac{x}{2}}}{2} \,. \tag{178h}$$

Thus

$$P_1 = 1 - \frac{e^{Y-\frac{x}{2}}}{2} Q(\sqrt{x}, 2\sqrt{Y}) + e^{-x}\int_0^Y v\,du \tag{178i}$$

or

$$P_1 = 1 - \frac{e^{Y-\frac{x}{2}}}{2} Q(\sqrt{x}, 2\sqrt{Y}) - e^{-x}\int_0^Y e^{-y} I_0(2\sqrt{xy})\,dy \,. \tag{178j}$$

The integral term in Eq.(178j) is just $1 - Q(\sqrt{2x}, \sqrt{2Y})$, and the final result is

$$P_1 = Q(\sqrt{2x}, \sqrt{2Y}) - \frac{e^{Y-\frac{x}{2}}}{2} Q(\sqrt{x}, 2\sqrt{Y}) \qquad Y > 0 \,. \tag{178k}$$

For $x = 0$, $Q(0,\beta) = e^{-\frac{\beta^2}{2}}$, and

$$P_1 = e^{-Y} - \frac{e^Y}{2}(e^{-2Y}) = \frac{e^{-Y}}{2} \tag{178l}$$

agreeing, as it should, with the result obtained from Eq.(145) by letting $N = 1$ and integrating. For $Y = 0$, $Q(\alpha,0) = 1$, and

$$P_1 = 1 - \frac{e^{-\frac{x}{2}}}{2} \tag{178m}$$

agreeing with Eq.(178a) when $Y = 0$.

The bias number for use with Eq.(178k) is obtained by

$$\Gamma_1 = \frac{0.693}{n} = \frac{e^{-Y_b}}{2} \tag{178n}$$

or

$$Y_b = 2.30 \log_{10} n - 0.327 \,. \tag{178o}$$

In Fig. 47 is shown a graph comparing Eq.(178k) with Eq.(23) for $n = 10^6$, where P is plotted as a function of x.

Though it might be possible to calculate the cumulative distributions for $N > 1$ by a method similar to that used for $N = 1$, it would be very tedious. Therefore resort is made to Gram-Charlier series, as before. The moments are directly obtainable from the characteristic function given in Eq.(143),

$$\nu_i = (-1)^i \frac{d^i}{dp^i} \left| \frac{e^{-Nx} \, e^{\frac{Nx}{p+1}}}{(1-p^2)^N} \right|_{p=0} \tag{179}$$

There seems to be no readily obtainable expression for ν_i in closed form. The first six moments obtained directly from Eq.(179) are:

$$\nu_1 = Nx \tag{180a}$$

$$\nu_2 = (Nx)^2 + 2Nx + 2N \tag{180b}$$

$$\nu_3 = (Nx)^3 + 6(Nx)^2 + 6Nx(N+1) \tag{180c}$$

$$\nu_4 = (Nx)^4 + 12(Nx)^3 + 12(Nx)^2(N+3) + 24(N+1) + 12N(N+1) \tag{180d}$$

$$\nu_5 = (Nx)^5 + 20(Nx)^4 + 20(Nx)^3(N+6) + 120(Nx)^2(N+2) + 60Nx(N+1)(N+2) \,. \tag{180e}$$

$$\nu_6 = (Nx)^6 + 30(Nx)^5 + 30(Nx)^4(N+10) + 120(Nx)^3(3N+10)$$
$$+ 180(Nx)^2(N+1)(N+6) + 360Nx(N+1)(N+2)$$
$$+ 120N(N+1)(N+2) \,. \tag{180f}$$

The corresponding central moments are:

$$\mu_2 = 2Nx + 2N = 2N(1+x) = \sigma^2 \tag{181a}$$

$$\mu_3 = 6Nx \tag{181b}$$

$$\mu_4 = 12(Nx)^2 + 24Nx(N+1) + 12N(N+1) \tag{181c}$$

$$\mu_6 = 120(Nx)^3 + 360(Nx)^2(N+3) + 360Nx(N+1)(N+2) + 120N(N+1)(N+2) \,. \tag{181d}$$

The central standard moments are:

$$\alpha_3 = \frac{3x}{\sqrt{2N}(1+x)^{\frac{3}{2}}} \tag{182a}$$

$$\alpha_4 = 3 + \frac{3(1+2x)}{N(1+x)^2} \tag{182b}$$

$$\alpha_6 \approx 15 + \frac{45\left(3x^2+3x+1\right)}{N(1+x)^3} \,. \tag{182c}$$

The coefficients of the series are, from Eq.(71):

$$c_3 = -\frac{x}{2\sqrt{2N}(1+x)^{\frac{3}{2}}} \tag{183a}$$

$$c_4 = \frac{1+2x}{8N(1+x)^2} \tag{183b}$$

$$c_6 = \frac{x^2}{16N(1+x)^3} \tag{183c}$$

The Gram-Charlier series for the probability density function is given by Eq.(93) where

$$t = \frac{y-\nu_1}{\sigma}, \quad \nu_1 = Nx, \; \sigma = \sqrt{2N(1+x)} \tag{184}$$

and the cumulative distribution is given by Eqs.(98) and (99). Figures 53 and 54, No.1, showing the comparison between the ordinary case and the composite case, were computed using the Gram-Charlier series developed above. There appears to be no significant difference in the probabilities of detection for N between 1 and 10. For N between 100 and 1,000, the composite case gives an effective signal-to-noise ratio about 1 db lower than the ordinary case.

ANOTHER APPROACH TO THE DETECTION CRITERIA-PROBABILITY THAT SIGNAL-PLUS-NOISE EXCEEDS NOISE ALONE

The method of setting a bias level and calling any signal-plus-noise or noise alone which exceeds this level a signal is not the only possible way of defining detection. Another method is based on asking what is the probability that any given signal will be larger than any noise pulse during a given interval of time[11]. The interval of time taken would logically be the false alarm time, as defined previously. In this time there will be $n/N = n'$ independent groups of noise pulses. If the probability that a single integrated group of signal-plus-noise pulses exceeds a single group of noise pulses is called $P_1(x,N)$, then the probability that the group of signal-plus-noise pulses exceeds *all* of the n' groups of noise pulses is simply

$$P = \left[P_1(x,N)\right]^{n'} \tag{185}$$

This probability is a little difficult to interpret properly. It means that if during the false alarm time a signal of strength x appears, it will have this probability of being larger than any noise pulse group appearing during the same time. The difficulty is how to pick out the largest signal over a period of time, and what to do when many signals are present. These are reasons why the earlier detection criteria are thought to be superior, since they provide clear answers for the above questions. The criteria presented above may be of special value, however, when a target is *known* to be present. Such is the case when a target is being automatically tracked, and one wishes to calculate the probability that it will be subsequently lost due to the noise exceeding the signal.

The probability density function for N signal-plus-noise pulses minus N noise pulses has been indicated in Eq.(161).

To obtain the probability that the sum of N signal-plus-noise pulses will be greater than N noise pulses it is only necessary to integrate Eq.(161) from 0 to ∞. It will be easier to obtain the probability that N noise pulses exceed N signal-

plus-noise pulses, however, since this requires the integral from $-\infty$ to 0, and an expression is available for $Y < 0$ in Eq.(167). Thus

$$P_{N>S+N} = \frac{e^{-\frac{Nx}{2}}}{(N-1)!}\int_{-\infty}^{0} dYe^{Y} \sum_{k=0}^{k=N-1} \frac{(N+k-1)!\,{}_1F_1\left(-k,N,-\frac{Nx}{2}\right)}{(N-k-1)!\,k!\,2^{N+k}}(-Y)^{N-k-1} \tag{186}$$

Now one substitutes z for $-Y$ and interchanges the summation and integration signs, obtaining

$$P = \frac{e^{-\frac{Nx}{2}}}{(N-1)!}\sum_{k=0}^{k=N-1}\left[\frac{(N+k-1)!\,{}_1F_1\left(-k,N,-\frac{Nx}{2}\right)}{(N-k-1)!\,k!\,2^{N+k}}\int_{0}^{\infty} e^{-z}\,z^{N-k-1}\,dz\right] \tag{187}$$

The integral is simply $(N-k-1)!$, and therefore

$$P = \frac{e^{-\frac{Nx}{2}}}{2}\sum_{k=0}^{k=N-1}\left[\frac{(N+k-1)!}{(N-1)!\,k!\,2^{N+k-1}}\,{}_1F_1\left(-k,N,-\frac{Nx}{2}\right)\right] \tag{188}$$

Or in terms of Laguerre polynomials, using Eq.(82),

$$P = \frac{e^{-\frac{Nx}{2}}}{2}\sum_{k=0}^{k=N-1} 2^{1-N-k}\,L_k^{N-1}\left(-\frac{Nx}{2}\right) \tag{189}$$

From Eq.(188) a more convenient form may be obtained by introducing a dummy index i and interchanging summation signs, leading eventually to

$$P = \frac{e^{-\frac{Nx}{2}}}{2}\sum_{i=0}^{i=N-1}\left[\frac{\left(\frac{Nx}{2}\right)^{i}}{i!\,(N+i-1)!}\sum_{k=i}^{k=N-1}\frac{(N+k-1)!}{(k-i)!\,2^{N+k-1}}\right] \tag{190}$$

The outside summation in Eq.(190) is obviously a polynomial in x of the N-1th degree and with N terms. It is rather curious to note that if one puts $i = 0$ in Eq.(190), the following identity results:

$$2^{N-1} \equiv \sum_{k=0}^{k=N-1}\frac{(N+k-1)!}{(N-1)!\,k!\,2^{k}} \tag{191}$$

In other words, the constant term in the polynomial is always unity.

The first few cases for low values of N are:

$$P_1 = \frac{1}{2} e^{-\frac{x}{2}} \tag{192a}$$

$$P_2 = \frac{1}{2} e^{-x}\left(1+\frac{x}{4}\right) \tag{192b}$$

$$P_3 = \frac{1}{2} e^{-\frac{3x}{2}}\left(1+\frac{9x}{16}+\frac{9x^2}{128}\right) \tag{192c}$$

$$P_4 = \frac{1}{2} e^{-2x}\left(1+\frac{29x}{32}+\frac{x^2}{4}+\frac{x^3}{48}\right) \tag{192d}$$

$$P_5 = \frac{1}{2} e^{-\frac{5x}{2}}\left(1+\frac{325x}{256}+\frac{575x^2}{1024}+\frac{4375x^3}{6144}+\frac{625x^4}{32,768}\right) . \tag{192e}$$

Obviously for N very large, these expressions rapidly become useless, and it is necessary to use the Gram-Charlier series of Eqs.(184) and (98). The lower limit Y is replaced by zero, giving for the series

$$P = \frac{1}{2}\left[1-\phi^{-1}(T)\right] + c_3\phi^2(T) - c_4\phi^3(T) - c_6\phi^5(T) ---- \tag{193}$$

where

$$T = +\frac{\nu_1}{\sigma} = x\sqrt{\frac{N}{2(1+x)}} \tag{194}$$

and c_3, c_4 and c_6 are given by Eqs.(183a-c). A graph of P as a function of x and N is shown in Fig.48. For very small values of P, more terms may be necessary in the series of Eq.(193).

USE OF CUMULANTS IN OBTAINING GRAM-CHARLIER SERIES COEFFICIENTS

It is often much simpler to obtain the cumulants for a given distribution function rather than the various moments. The cumulants may be defined by

$$\kappa_i = (-1)^i\left(\frac{d^i}{dp^i} \log_e C\right)_{p=0} \tag{195}$$

where C is the characteristic function of the given probability density function (see pages 61-65 of Kendall[6]). The cumulants, except the first, are invariant with respect to a change of origin. Also, for the distribution of the sum of N variates, it is only necessary to multiply every cumulant by N, as is evident from the defining Eq.(195). The coefficients of the Gram-Charlier series in terms of cumulants are given on page 149 of Kendall. The cumulants in standard measure may be defined as

$$K_i = \frac{\kappa_i}{\sigma^i} \quad . \tag{196}$$

In terms of standard cumulants, the coefficients of the series are:

$$c_0 = 1 \quad c_1 = c_2 = 0 \tag{197a}$$

$$c_3 = \frac{K_3}{3!} \tag{197b}$$

$$c_4 = \frac{K_4}{4!} \tag{197c}$$

$$c_6 = \frac{1}{6!}\left(K_6 + 10K_3^2\right) \quad . \tag{197d}$$

The first term in Eq.(197d), K_6, is omitted in the 0,3,4,6 approximation.

Consider the square law case where, from Eq.(35),

$$C = \frac{e^{-x} e^{\frac{x}{p+1}}}{p+1} \tag{198}$$

$$\log_e C = -x + \frac{x}{p+1} - \ln (p+1) \quad . \tag{199}$$

From Eq.(195),

$$\kappa_i = (i-1)!(ix+1) \qquad i \neq 1 \quad . \tag{200}$$

For N variates,

$$\kappa_i = N(i-1)!(ix+1) \quad . \tag{201}$$

and

$$K_i = \frac{\kappa_i}{\sigma^i} = \frac{(i-1)!(ix+1)}{N^{\frac{i}{2}-1}(2x+1)^{\frac{i}{2}}} \quad . \tag{202}$$

In particular,

$$K_3 = \frac{2(3x+1)}{N^{1/2}(2x+1)^{\frac{3}{2}}} \tag{203a}$$

$$K_4 = \frac{6(4x+1)}{N(2x+1)^2} \tag{203b}$$

and it is at once evident that c_3, c_4, and c_6 obtained from Eqs.(197b-d) are identical to the values given by Eqs.(92c-e) by means of a much longer process.

In the case of a composite pulse of signal-plus-noise minus noise, the characteristic function is given by Eq.(142) and

$$\log_e C = -x + \frac{x}{p+1} - \log_e (1-p^2) \ . \tag{204}$$

Again by means of Eq.(195) it is easy to derive, for N variates,

$$\begin{aligned} \kappa_i &= N(i-1)!\,[ix+2] \qquad i \text{ even} \\ &= N(i-1)!\,(ix) \qquad i \text{ odd}, \neq 1 \end{aligned} \tag{205}$$

or

$$\kappa_i = N(i-1)!\left[ix+1+(-1)^i\right] \tag{206}$$

and

$$K_i = \frac{\kappa_i}{\sigma^i} = \frac{(i-1)!\left[ix+1+(-1)^i\right]}{N^{\frac{i}{2}-1}\left[2(x+1)\right]^{\frac{i}{2}}} \ . \tag{207}$$

Special cases are:

$$K_3 = \frac{6x}{N^{1/2}\left[2(x+1)\right]^{\frac{3}{2}}} \tag{208a}$$

$$K_4 = \frac{3(2x+1)}{N(x+1)^2} \tag{208b}$$

and again by Eqs.(197b-d) the coefficients are seen to be the same as given by Eqs.(183a-c).

In a case such as the linear one where the characteristic function cannot be obtained, the cumulants are still useful and may be found from the moments ν_i by means of the formulae at the bottom of page 63 of Kendall. The first few are:

$$\kappa_1 = \nu_1 \tag{209a}$$

$$\kappa_2 = \nu_2 - \nu_1^2 \tag{209b}$$

$$\kappa_3 = \nu_3 - 3\nu_2\nu_1 + 2\nu_1^3 \tag{209c}$$

$$\kappa_4 = \nu_4 - 4\nu_3\nu_1 - 3\nu_2^2 + 12\nu_2\nu_1^2 - 6\nu_1^4 \; . \tag{209d}$$

The κ_i are now obtained by multiplying by N and dividing by σ^i. The coefficients are then obtained as before by Eqs.(197a-d).

BEST POSSIBLE DETECTOR LAW

It is of considerable importance to know whether there may be some detector law which will give results which are appreciably better than the linear or square law cases which have already been considered.

The problem may be stated as follows:

These are available N samples

$$v_1, \; v_2 \; ---- \; v_N$$

which, it is assumed, are known to have come from either the distribution

$$dP_1 = ve^{-\frac{v^2}{2}} dv \tag{210}$$

or the distribution

$$dP_2 = ve^{-\frac{v^2+a^2}{2}} I_0(av)\, dv \tag{211}$$

the former being the distribution of the envelope of noise alone, and the latter the distribution of signal-plus-noise.

The probability that all of the variates $v_1 ---- v_N$ came from the second distribution is simply

$$dP_{N2} = dP_2(v_1)\, dP_2(v_2) \; ---- \; dP_2(v_N) \tag{212}$$

whereas the probability that they all came from the first distribution is

$$dP_{N1} = dP_1(v_1)dP_1(v_2) \text{ ---- } dP_1(v_N) \tag{213}$$

The ratio of dP_{N2} to dP_{N1} is the best measure of the likelihood that all the variants came from the signal-plus-noise distribution. It can be shown that any monotonic function of this ratio gives an equally good significance test. One arbitrarily picks a constant which the ratio must exceed to say that it shows that the variants came from the signal-plus-noise distribution. *This constant determines the false alarm time.*

Taking the ratio of Eq.(213) to Eq.(212) and substituting values from Eqs.(210) and (211) gives

$$\frac{dP_{N2}}{dP_{N1}} = \frac{\prod_{i=1}^{i=N} v_i e^{-\frac{v_i^2+a^2}{2}} I_0(av_i)}{\prod_{i=1}^{i=N} v_i e^{-\frac{v_i^2}{2}}} \tag{214}$$

or

$$e^{-\frac{a^2}{2}} \prod_{i=1}^{i=N} I_0(av_i) \geqslant \lambda \tag{215}$$

where λ is the constant which determines the false alarm time.

Taking the log of both sides of Eq.(215) gives

$$\sum_{i=0}^{i=N} \log_e \, I_0(av_i) \geqslant \log_e \, \lambda + \frac{a^2}{2} \tag{216}$$

Note that *nothing* has been said in the foregoing discussion about integration. Now, however, Eq.(216) says that the best thing to do is take the log of I_0 of each variate, add these functions for each variate, and require the sum to exceed a certain value. Clearly this calls for a detector and integrator which has the combined law

$$y = \log \, I_0(av) \tag{217}$$

The meaning of this result is really quite remarkable (at least to one who is not a statistician). It says, in effect, that by having the sum only of N variates which have been subjected to the law $y = \log \, I_0(av)$, one has as much useful information as if the individual values of each of the variates were known (as far as determining to which distribution the variates belong)*.

* If the two distribution functions to be distinguished are normal, then the simple sum of the N variates, or the mean, is the best criterion. In other words, a linear law would be the best if the envelopes of noise and signal-plus-noise were normally distributed.

Suppose that the signal strength is very small (which would make N large for any reasonable probability of detection). Then $I_0(av) \approx 1 + a^2v^2/4$ and

$$y = \log I_0(av) \approx \log \left(1+\frac{a^2v^2}{4}\right) \approx \frac{a^2v^2}{4} \tag{218}$$

In this case, the square law is seen to be the best possible choice. If, on the other hand, the signal strength is large, $I_0(av) \approx e^{av}/\sqrt{2\pi av}$ and

$$y = \log I_0(av) \approx \log \frac{e^{av}}{\sqrt{2\pi av}} \approx av - \frac{1}{2}\log 2\pi av \approx av \tag{219}$$

Thus, for large signals (usually small N) the linear law is best.

It should be pointed out that the results for the two extreme cases, square and linear law, are not very different (see Fig. 42), and in practice a linear detector would usually be preferred on account of its relative immunity to saturation by large signals.

In the case of a human operator it is difficult to say what law is used in the process of integration. Thus if a linear detector were used in the receiver, it is conceivable that the operator might mentally take the sum of the squares in his integration process, with a net over-all square law effect.

SIGNAL-PLUS-NOISE MINUS NOISE — LINEAR LAW

This case is of special interest because of the method which must be used in obtaining the solution. Since the characteristic function for the linear case cannot be found, it is necessary to determine the moments for a composite variate directly from the moments for the signal-plus-noise distribution and those for the noise distribution alone.

Using a double subscript notation, in which the first index represents the number of the distribution function and the second index represents the order of the moment, the following formulae can be derived at once by successive differentiations of the product of the characteristic functions of the individual distribution functions:

$$\nu_1 = \nu_{11} + \nu_{12} \tag{220a}$$

$$\nu_2 = \nu_{12} + \nu_{22} + 2\nu_{11}\nu_{22} \tag{220b}$$

$$\nu_3 = \nu_{23} + \nu_{13} + 3(\nu_{11}\nu_{22} + \nu_{12}\nu_{21}) \tag{220c}$$

$$\nu_4 = \nu_{14} + \nu_{24} + 4(\nu_{11}\nu_{23} + \nu_{21}\nu_{13}) + 6\nu_{12}\nu_{22} \tag{220d}$$

$$\nu_6 = \nu_{16} + \nu_{26} + 6(\nu_{25}\nu_{11} + \nu_{15}\nu_{21}) + 15(\nu_{12}\nu_{24} + \nu_{22}\nu_{14}) + 20\nu_{13}\nu_{23} \tag{220e}$$

The first set of moments are those for one variate of signal plus noise given by Eqs.(109) and (110a-d). The second set of moments are for one *negative* variate of noise alone. These are simply obtained from the first set of moments by putting $x = 0$ and multiplying the odd moments by -1.

The details will not be given, since the results bear the same relation in general to the square law case as they do when a noise variate is not subtracted from each signal-plus-noise variate.

USE OF SO-CALLED DETECTION CRITERIA

Lawson and Uhlenbeck have made use of a quantity which is the shift in average value of a distribution of signal-plus-noise from that of noise alone divided by the standard deviation of noise alone, which they call the detection criterion. In symbolic form

$$k = \frac{\nu_{1S+N} - \nu_{1N}}{\sigma_N} \quad . \tag{221}$$

This quantity is also called the deflection criterion, and it is implied that it must be of the order of unity or greater to have a reasonable probability of detection.

For the square law detector, using the results of Eqs.(81a-b) and (85c), the criterion becomes

$$k = x\sqrt{N} \tag{222}$$

and for the linear detector

$$k = \frac{x\sqrt{N}}{2\sqrt{\frac{4}{\pi} - 1}} = 0.957x\sqrt{N} \tag{223}$$

assuming x to be small.

The object of these criteria is to show the variation in necessary signal-to-noise ratio as a function of the number of pulses integrated. The results for k in Eqs.(222) and (223) may be derived rigorously from the basic distribution equations if the central limit theorem is assumed to hold and for probability of detection equal to 0.50.

However, it is found from the actual results presented in No.1, Figs.1-50, that the square root of N law given by the detection criteria is not closely followed, even for N as large as 1,000. If a law of the form

$$k = xN^{\theta} \tag{224}$$

is assumed, the exponent θ may be obtained from the data of Figs.1-50, No.1. The results are given in Figs.55 and 56, No.1. It is seen that θ goes from 1.0 at N = 1 to around 0.75 at N = 1,000. As pointed out earlier (page 182), θ = 1 for coherent integration.

It has been said that the $N^{1/2}$ law seems to fit observed data fairly well. It is the belief of the author that this is a coincidence that arises from the fact that the losses due to nonlinear integration by cathode ray tubes, and human operator losses, tend to just about equal the difference between N^{θ} and $N^{1/2}$, so that the $N^{1/2}$ law actually seems to fit the *observed* data.

It is rather interesting to note that if the detector law is assumed to be of the form $y = v^n$, the detection criterion turns out to be

$$k = \frac{n\left(\frac{n}{2}!\right)}{2\sqrt{n! - \left(\frac{n}{2}!\right)^2}}\, x\sqrt{N} \quad . \tag{225}$$

A graph of this function shows a very broad maximum of 1 at n = 2. Thus this is a special case, showing that for large N the square law is the best of the particular class of functions v^n. This is not as general as the proof on page 210 which shows that the square law is the best of *all possible functions* for small x.

COLLAPSING LOSS – INTEGRATION OF GREATER NUMBER OF NOISE VARIATES THAN OF SIGNAL-PLUS-NOISE VARIATES

In many radar applications, an additional number of noise variates are integrated along with a given number of signal-plus-noise variates. Such is the case when three-dimensional data are compressed onto a two-dimensional presentation, or with a C scope where range is not shown. The loss so occasioned is called a collapsing loss[39]. An effect of the same kind is caused if the spot of a cathode ray tube indicator moves less than its diameter in a pulse length[11]. Again, if the video bandwidth is narrow compared with the IF bandwidth, the same sort of thing happens. All three effects are handled by assuming a given collapsing ratio, ρ, which is defined by

$$\rho = \frac{M + N}{N} \tag{226}$$

where

N = number of signal-plus-noise variates integrated

M = number of effective additional noise variates integrated.

In the case of loss caused by low writing speed of the cathode ray beam, the effective collapsing ratio is given approximately by

$$\rho_{\text{eff}} = \frac{d + s\tau}{s\tau} \tag{227}$$

where

d = spot diameter

s = writing speed

τ = pulse length.

Where the loss is caused by a video amplifier, the equivalent defining equation is

$$\rho_{\text{eff}} = \frac{B_{if}+B_v}{B_v} \tag{228}$$

where

B_{if} = IF bandwidth (or total combined RF and IF bandwidth where RF amplification is used)

B_v = video bandwidth.

Mathematically, the treatment necessary to take account of M extra noise variates is rather simple. It is only necessary to multiply the characteristic function for N signal-plus-noise variates by the characteristic function for M noise-alone variates. In the square law case, this results in

$$C_N = \frac{e^{-Nx}\, e^{\frac{Nx}{p+1}}}{(p+1)^{N+M}} = \frac{e^{-N\rho\left(\frac{x}{\rho}\right)}\, e^{\frac{N\rho\left(\frac{x}{\rho}\right)}{p+1}}}{(p+1)^{N\rho}} \quad . \tag{229}$$

It is apparent, by comparison with Eq.(36), that the results obtained for $\rho = 1$ can be used directly to obtain results for any ρ.

Care must be taken in obtaining the bias level, however. Without the M extra noise variates, the relation $n' = n/N$ is used to find the required signal-to-noise ratio, x. With the added noise variates, the number of groups of pulses integrated may or may not remain the same. In the case of *video mixing*, where the output of two independent radars is superimposed on the same indicator, the number of groups of pulses integrated is constant, which means that n' is constant.

In the other cases where the loss is caused by narrow video amplifiers, collapsing of coordinates, or slow writing speed, the number of independent groups of pulses integrated is reduced by the factor ρ_{eff} so that n remains constant as is easily seen from the equations

$$n = n'N \qquad \text{(no loss)} \tag{230}$$

$$n = (\rho n')(M+N) = n'N \qquad \text{(with loss)}. \tag{231}$$

The collapsing loss is defined as

$$L_c = 10 \log_{10} \frac{x_2}{x_1} \tag{232}$$

where x_2 is the required signal-to-noise ratio with M extra noise variates, and x_1 is the signal-to-noise ratio required with no extra noise variates, such that

the probability of detection is the same in both cases. This fixed probability level will usually be taken as 0.90.

The procedure, after finding x_1, is to get the required bias from either Fig. 8 or 9, depending on whether n or n' is held constant, using ρN as the number of variates. From the cumulative distribution functions graphed in Figs. 13 to 32, the value of x_2 is found by multiplying the finding x for ρN variates to give $P = 0.90$ and multiplying this value of x by ρ. The reason for multiplying by ρ is apparent on referring to Eq.(229).

The results of the calculation are shown in Figs. 49 to 52 where L is plotted as a function of N for $P = 0.90$ and $N = 10^6$. Also given are curves of θ_c defined by

$$\frac{x_2}{x_1} = N^{\theta_c} . \tag{233}$$

It has commonly been said that θ_c should be 1/2[36],[35]. This statement is sometimes derived from the detection criterion given on page 212.

From Fig. 57 it is seen that if n' is constant, θ does approach 1/2 as $N \to \infty$. However, θ is much smaller for reasonably small N. In the case of n constant, the square root law is not even approached as an asymptote.

It was found that the values of L and θ are only slightly dependent on the original values of n and P.

ANTENNA BEAM SHAPE LOSS

It has so far been assumed that the antenna pattern was flat over the half-power beamwidth and zero elsewhere. In any practical case the beam shape may usually be approximated by a Gaussian curve which will hold fairly well out to ± the beamwidth from the point of maximum gain. In the case of a searchlighting antenna, the returned pulses will all fall at the same place in the beam, and if this does not happen to fall at the maximum of the beam, the loss may easily be taken into account by modifying the expression for gain used in Eq.(9), No.1 for calculating R_0 such that

$$G = G_{\max}\, e^{-4\ln 2\left(\frac{\theta_a^2}{B_a^2}+\frac{\theta_e^2}{B_e^2}\right)} \tag{234}$$

where

- θ_a = azimuth angle between target and antenna axis
- θ_e = elevation angle between target and antenna axis
- B_a = half-power azimuth beamwidth
- B_e = half-power elevation beamwidth

If the antenna is scanning, the problem is entirely changed because the successive returned pulses will be of different magnitude. It is obvious that as the antenna scans past a target, pulses should be integrated out to some point where the principle of diminishing returns sets in. It is not too difficult to determine this point and to calculate the loss occasioned due to the beam shape as compared with the ideal case[43]. A complete treatment which covers the general case of delay

of the received pulse relative to the transmitted pulse, off axis in elevation while scanning in azimuth, and random orientation of the pulse pattern relative to the antenna pattern is quite involved. However, the solution of some special cases has shown the general character to be expected of the results.

The integration of pulses should be carried to about 1.1 times the half-power beamwidth. This figure is practically independent of the signal strength (range) and the number of pulses per half-power beamwidth. When the optimum number of pulses are integrated there will be an average loss over the ideal case which assumes constant gain between the half-power points. This loss is in the neighborhood of 1.5 db and does not depend much on signal strength or number of pulses per half-power beamwidth. Since this loss is so small it was not considered worth while to reproduce all the detailed calculations here.

It should be mentioned that special care is necessary when one considers rates of antenna scanning so fast that about only 1 hit per beamwidth is obtained. In this case it may be expedient to make the receiving antenna lag the transmitting antenna to compensate for the time of travel of the pulse, or to step-scan, that is, move the antenna in discrete steps rather than continuously.

In order to calculate the probability of detection in any case where the successive returned pulses have different signal strengths, it is necessary to obtain the over-all characteristic function by multiplying the characteristic functions for each pulse. Using this method it is not difficult to work out the needed results in any particular case.

LIMITING LOSS

If limiting occurs anywhere in the receiver, the probability of detection will be lowered, everything else being held constant. The video amplifier is the first place where limiting will probably occur. Let the limiting ratio be defined as the ratio of the limit level to the R.M.S. noise level. Limiting can then be represented mathematically by replacing the probability density function at the detector output by an equivalent function below the limit level, and a delta function at the limit level having an area equal to all of the area of the original function to the right of the limit level. The moments can be calculated for these new functions (noise alone and signal-plus-noise), and the probability of detection found by use of the Gram-Charlier series as usual. The calculations are quite tedious and will not be reproduced here. The main conclusions are that if the number of pulses integrated is large, the limiting loss is only a fraction of a db if the limiting ratio is as large as 2 or 3, but if only one or two pulses are integrated the limiting ratio must be in the neighborhood of 10 to prevent a serious loss.

Limiting in the output of the integrator can also cause a loss, but this loss is small compared to the loss caused by limiting of the individual pulses in most practical cases.

EFFECT OF SIGNAL INJECTION ON PROBABILITY OF DETECTION

It has been proposed that the minimum detectable signal can be decreased by the injection of an RF or IF carrier voltage that adds linearly to the received

echo and the receiver noise[51]. The theory is that the total signal will then be large compared with the noise, and thus the so-called modulation suppression that occurs in the process of detection with small signals will be eliminated.

In such a process, the coherence of the injected signal with the received echo must be taken into account. If the target is moving, then the successive received pulses may be considered to be random in phase, so that the injected signal will necessarily be noncoherent with the echo. Analysis has shown that in this case the probability of detection decreases continuously as the magnitude of the injected signal increases, assuming a linear or square law detector. However, it can be shown that the best possible detector law starts to change radically as soon as the injected signal strength becomes comparable to noise. The analysis of probability of detection when the detector function is altered to take into account the injected signal has not been completed. Preliminary estimates indicate that there will be only a small decrease in sensitivity in this case.

It might be imagined that coherence could be obtained in a system using only one hit per target but having, say, 20 separate receiver channels with 20 separate injection oscillators having phases spaced 12 degrees apart. Thus, the return echo would be nearly coherent with some one of the channels. Theoretically, the improvement in this channel would be about 1 db. However, even this improvement would be just offset by the increased false alarm number due to the multiple channels, so that the over-all system improvement would be nil. It seems that there is no way to increase system sensitivity to moving targets by signal injection.

There is some possibility of increasing sensitivity for stationary targets by coherent signal injection, but it is difficult to imagine a practical situation where such a method would be of any use.

PROBABILITY OF DETECTION WITH MOVING TARGET INDICATION SYSTEMS

The analysis of the probability of detection for MTI systems is quite complicated. It depends on the type of receiver (lin-log limiting or IAGC), the type of detector, and the characteristics of the storage device used. For a nonfluctuating clutter and no scanning noise, the effect of the clutter with or without the addition of a coherent oscillator is much the same as that of the injected carrier discussed in the previous section. If a suitable detection system is used, the sensitivity may be reduced by a small amount, due to the addition of the coho, perhaps by 1 to 3 db.

The sensitivity of an MTI system for high probabilities of detection is further reduced due to the fact that the target may be moving at a speed differing from one of the so-called optimum speeds. This effect is quite complicated and is similar to that caused by a random variation of the cross section of a target with aspect. A method of quantitatively treating these problems has been developed and will be presented in detail in a future report.

If there is a fluctuation component in the clutter, due either to the movement of the clutter itself or to the scanning of the antenna, the effect will be to increase the amount of noise at the receiver input. This can be taken into account by an appropriate adjustment in the value of the noise figure of the receiver that will change R_0 by the correct amount.

TABLES OF THE DERIVATIVES OF THE ERROR FUNCTION

In order to make efficient use of Gram-Charlier series, it is necessary to have a good table of the derivatives of the error integral (the ϕ functions of Eq. 62). No satisfactory table was in existence at the time this report was written. Typical of the available tables[3] were

Fry[5]	$n = 1(1)6,\ x = 0(.1)4$	5 decimals
Jorgensen	$n = 1(1)6,\ x = 0(.01)4$	7 decimals

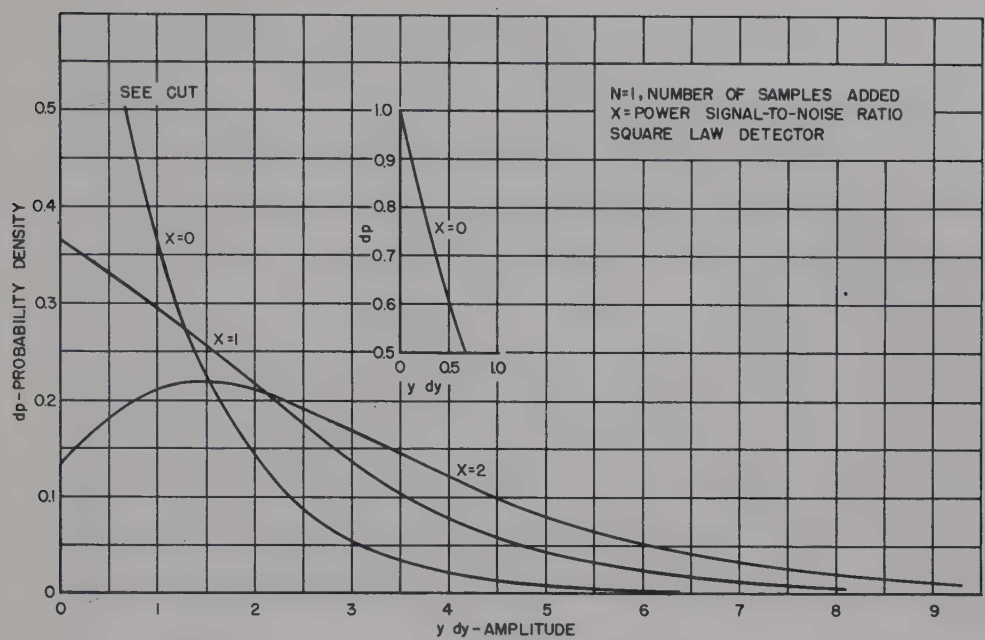
and an unpublished table of the W.P.A., giving

$n = 1(1)14,\ x = 0(.1)8.4$ 20 decimals

RAND therefore decided to calculate a suitable table with the aid of its IBM equipment. This has resulted in a table of Hermite polynomials, as well as in the derivatives of the error integral, giving

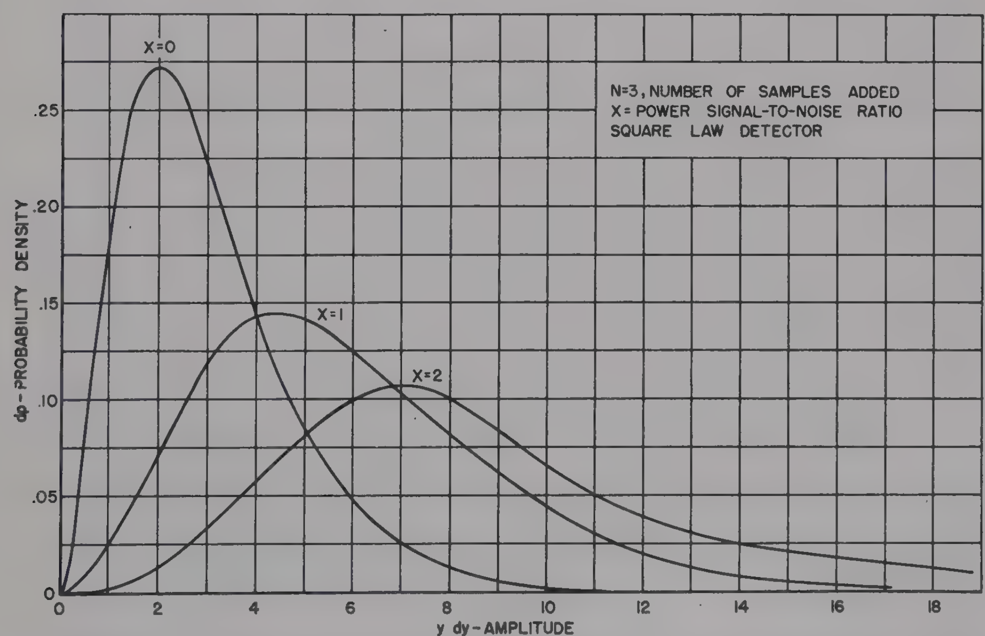
$n = 1(1)10,\ x = 0(.01)12.0$ 6 significant figures

A limited number of these tables are available at the present time. (RAND Document D-350, *A Table of Hermite Polynomials and the Derivatives of the Error Function.*)



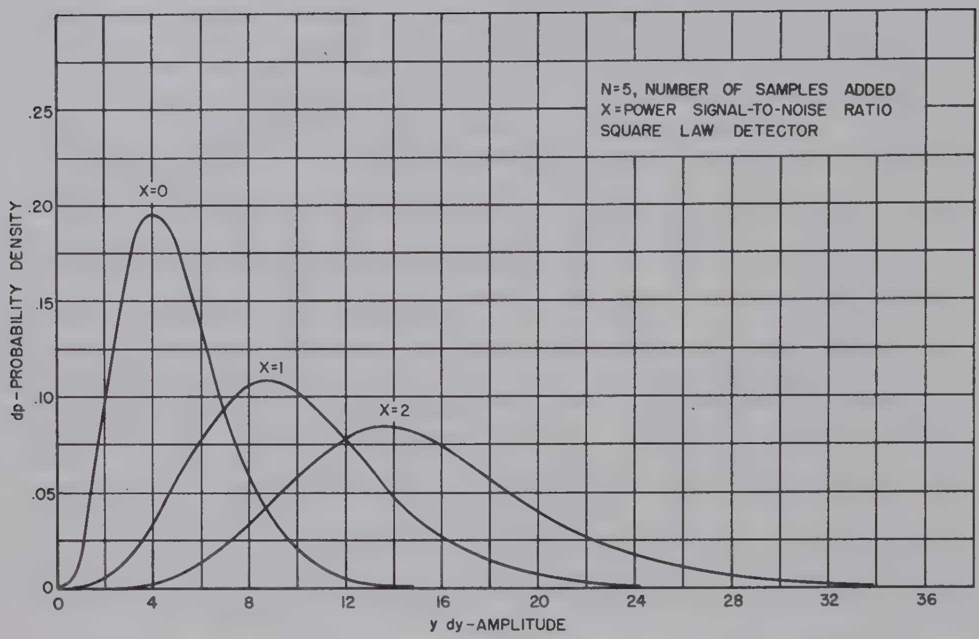
PROBABILITY DENSITY FUNCTIONS FOR ENVELOPE OF SINE WAVE PLUS NOISE

FIG. 1



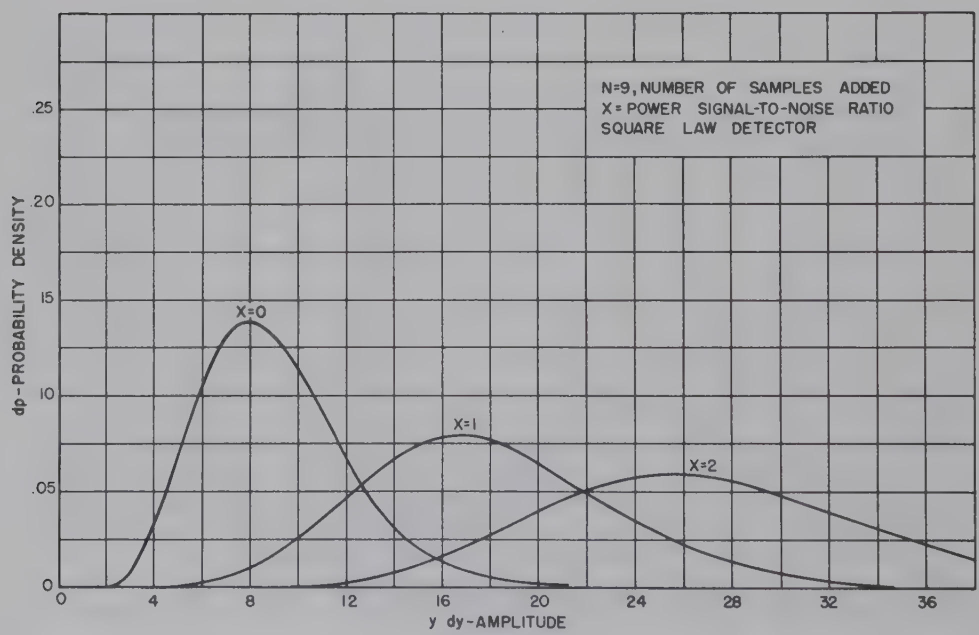
PROBABILITY DENSITY FUNCTIONS FOR ENVELOPE OF SINE WAVE PLUS NOISE

FIG. 2



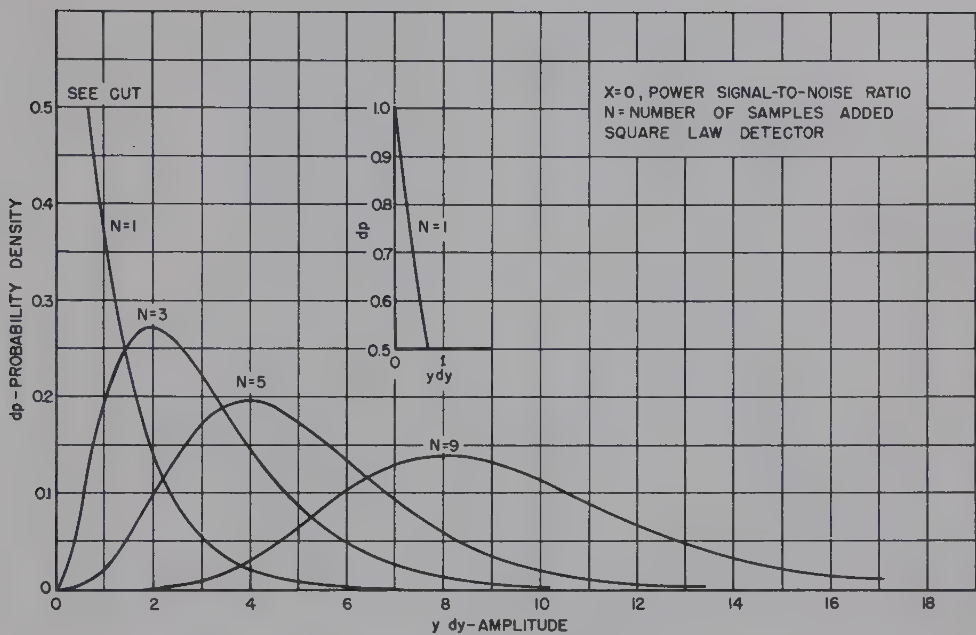
PROBABILITY DENSITY FUNCTIONS FOR ENVELOPE OF SINE WAVE PLUS NOISE

FIG. 3



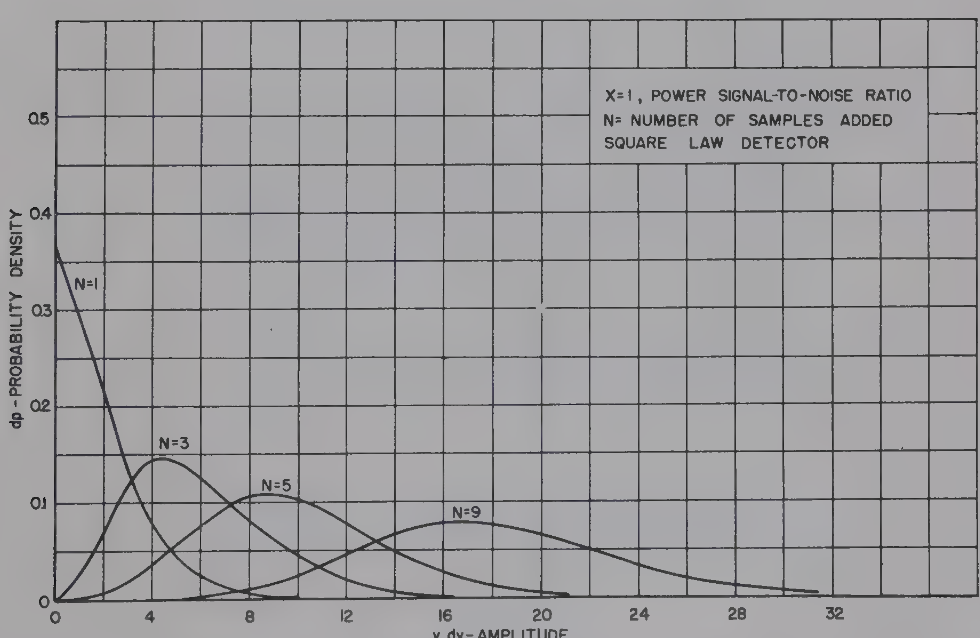
PROBABILITY DENSITY FUNCTIONS FOR ENVELOPE OF SINE WAVE PLUS NOISE

FIG. 4



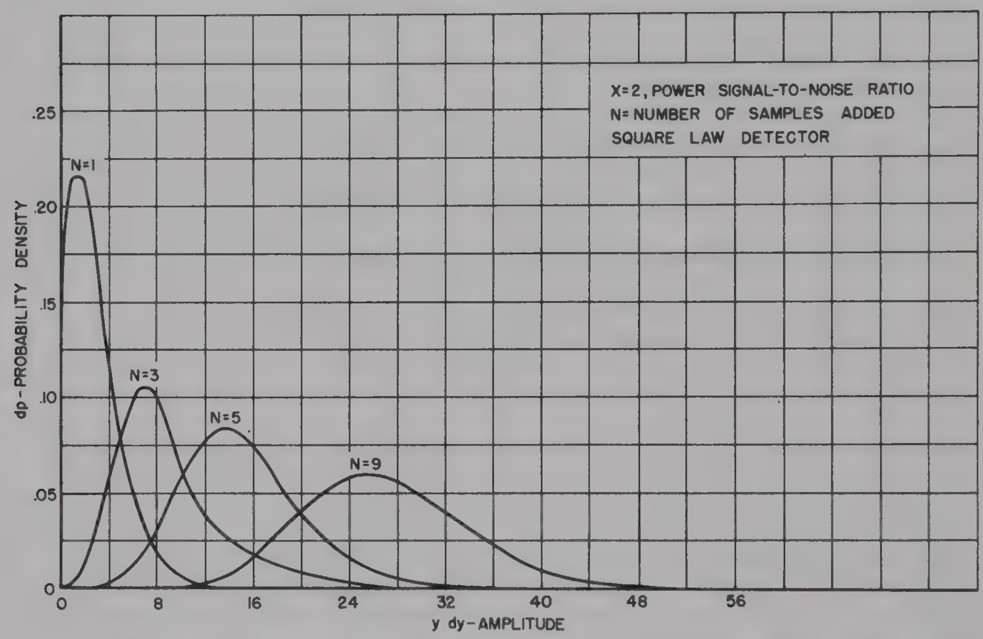
PROBABILITY DENSITY FUNCTIONS FOR ENVELOPE OF SINE WAVE PLUS NOISE

FIG. 5



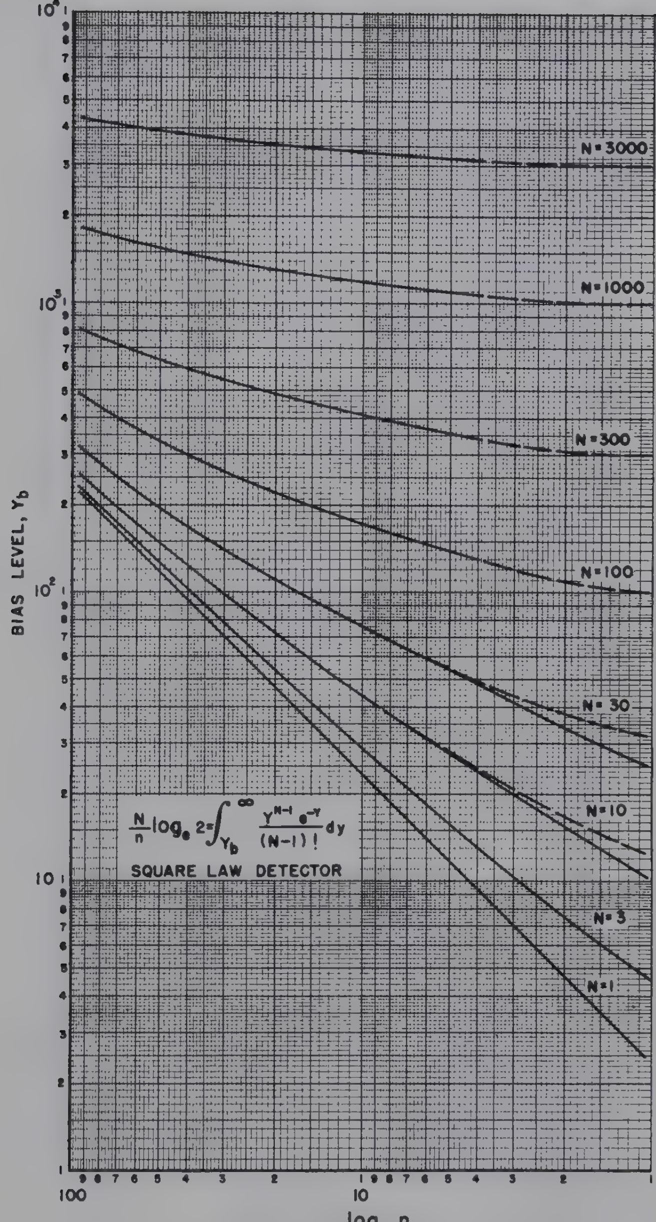
PROBABILITY DENSITY FUNCTIONS FOR ENVELOPE OF SINE WAVE PLUS NOISE

FIG. 6



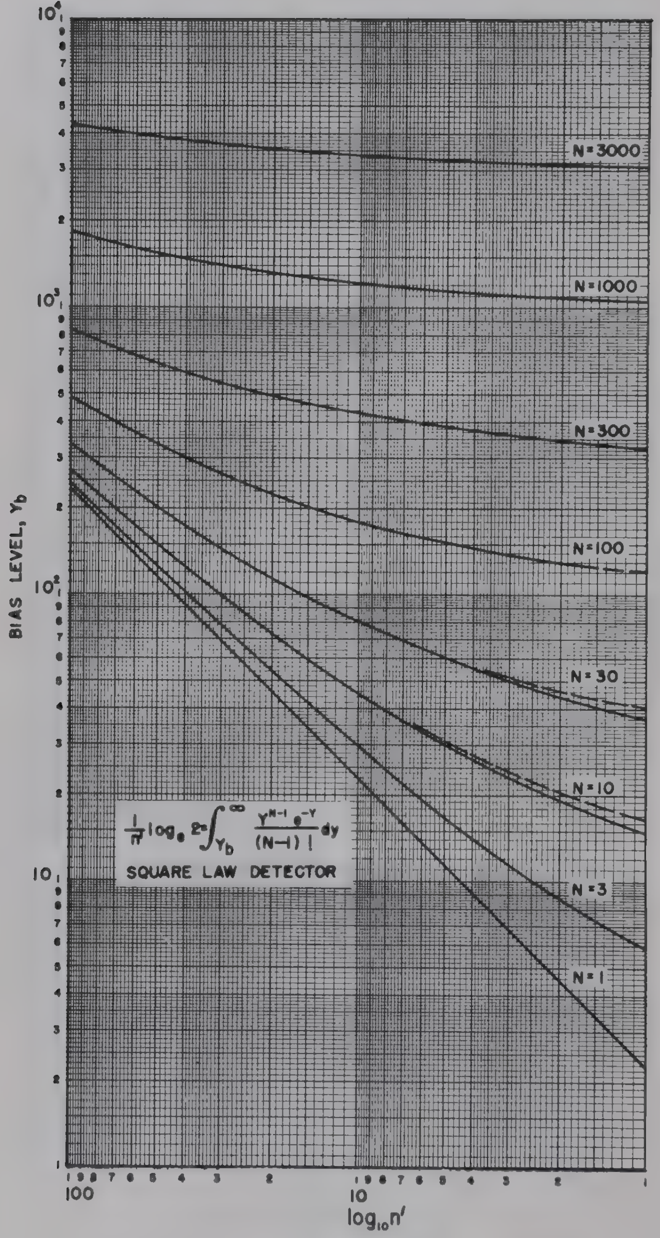
PROBABILITY DENSITY FUNCTIONS FOR ENVELOPE OF SINE WAVE PLUS NOISE

FIG. 7



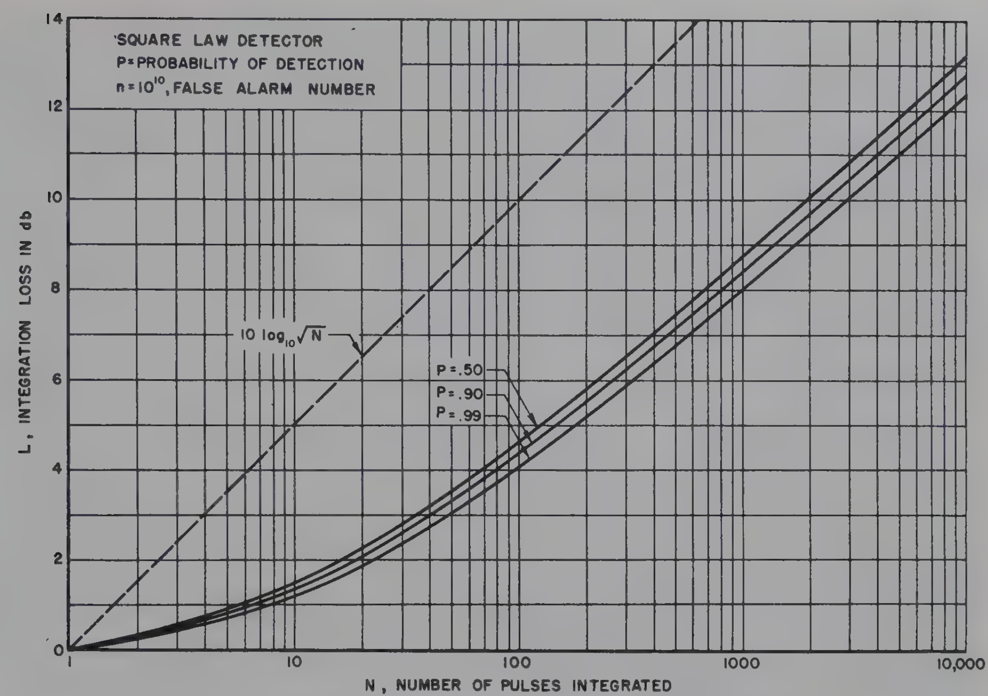
BIAS LEVEL AS A FUNCTION OF NUMBER OF PULSES INTEGRATED AND FALSE ALARM NUMBER

FIG. 8



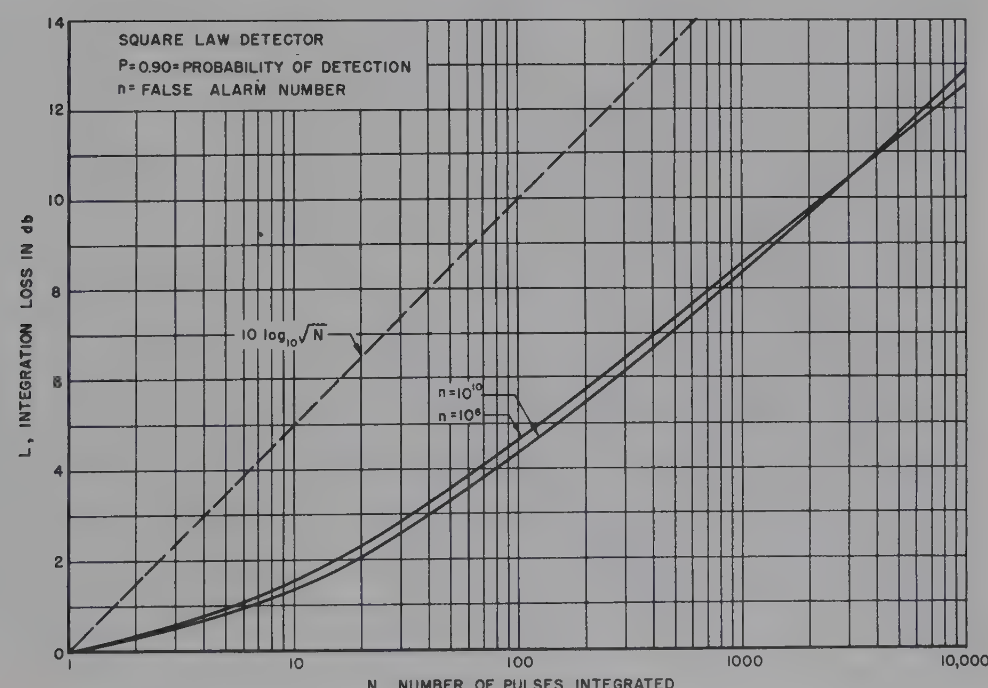
BIAS LEVEL AS A FUNCTION OF NUMBER OF PULSES INTEGRATED AND FALSE ALARM NUMBER

FIG. 9



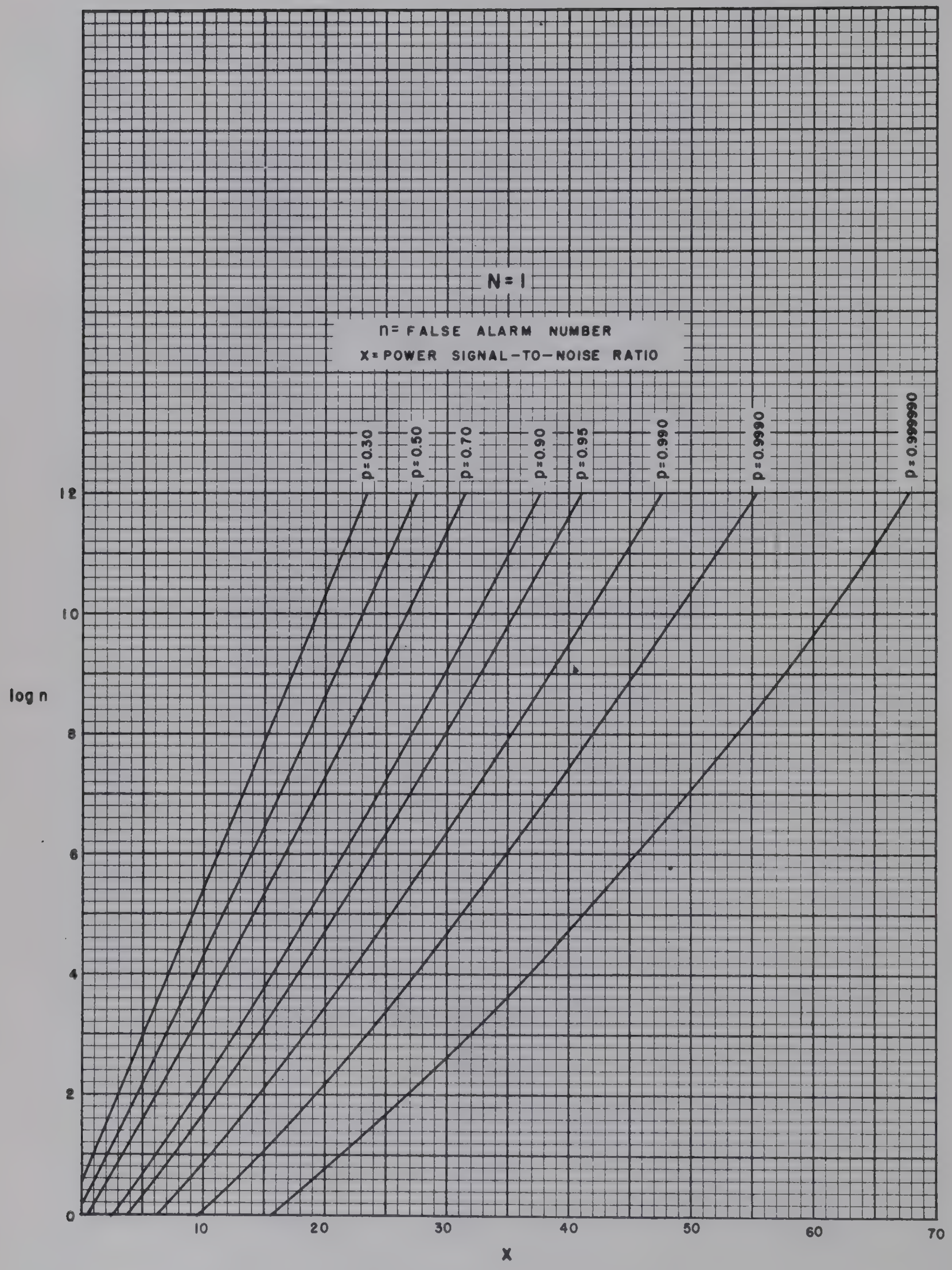
INTEGRATION LOSS, NON-COHERENT vs COHERENT

FIG.10



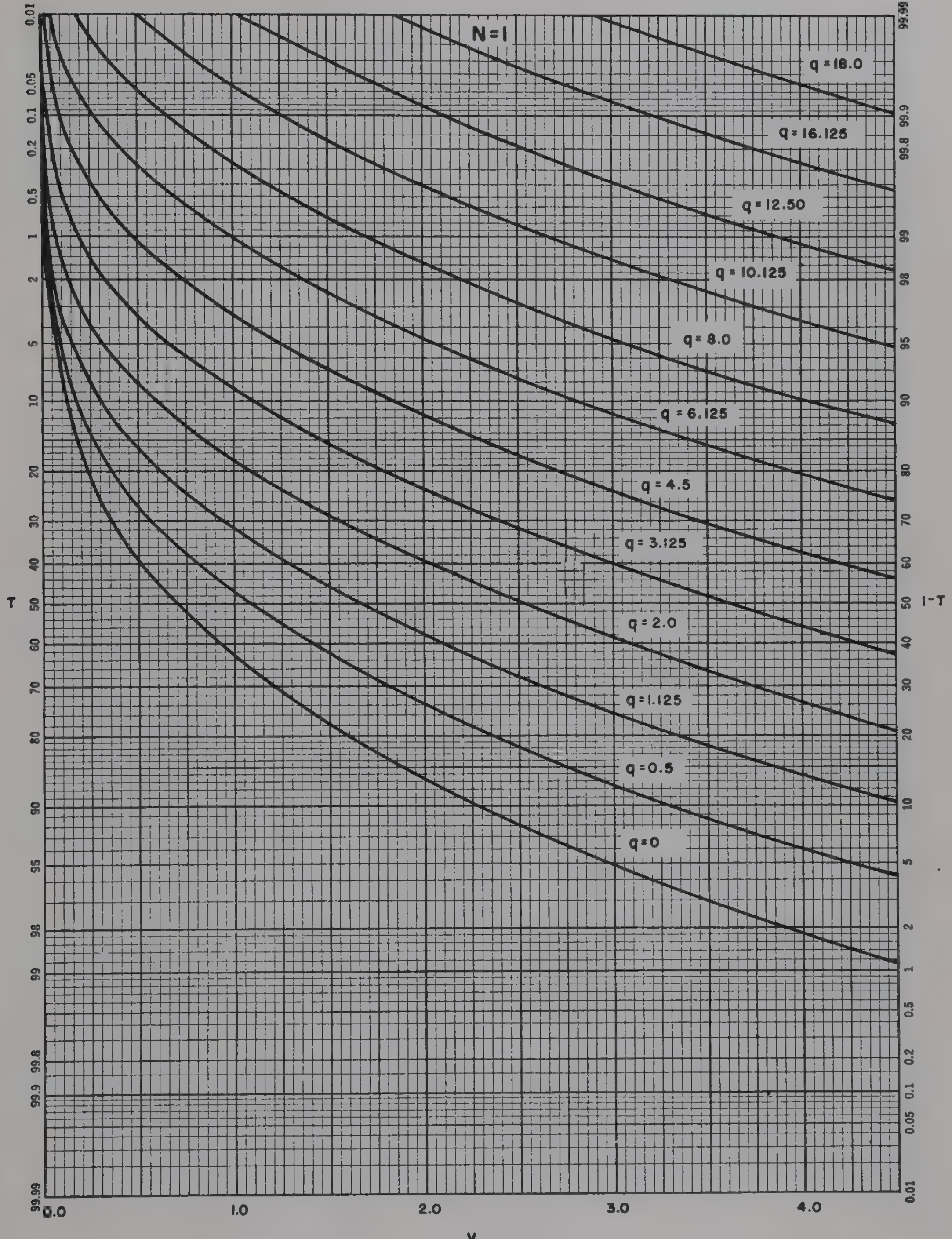
INTEGRATION LOSS, NON-COHERENT vs COHERENT

FIG.11



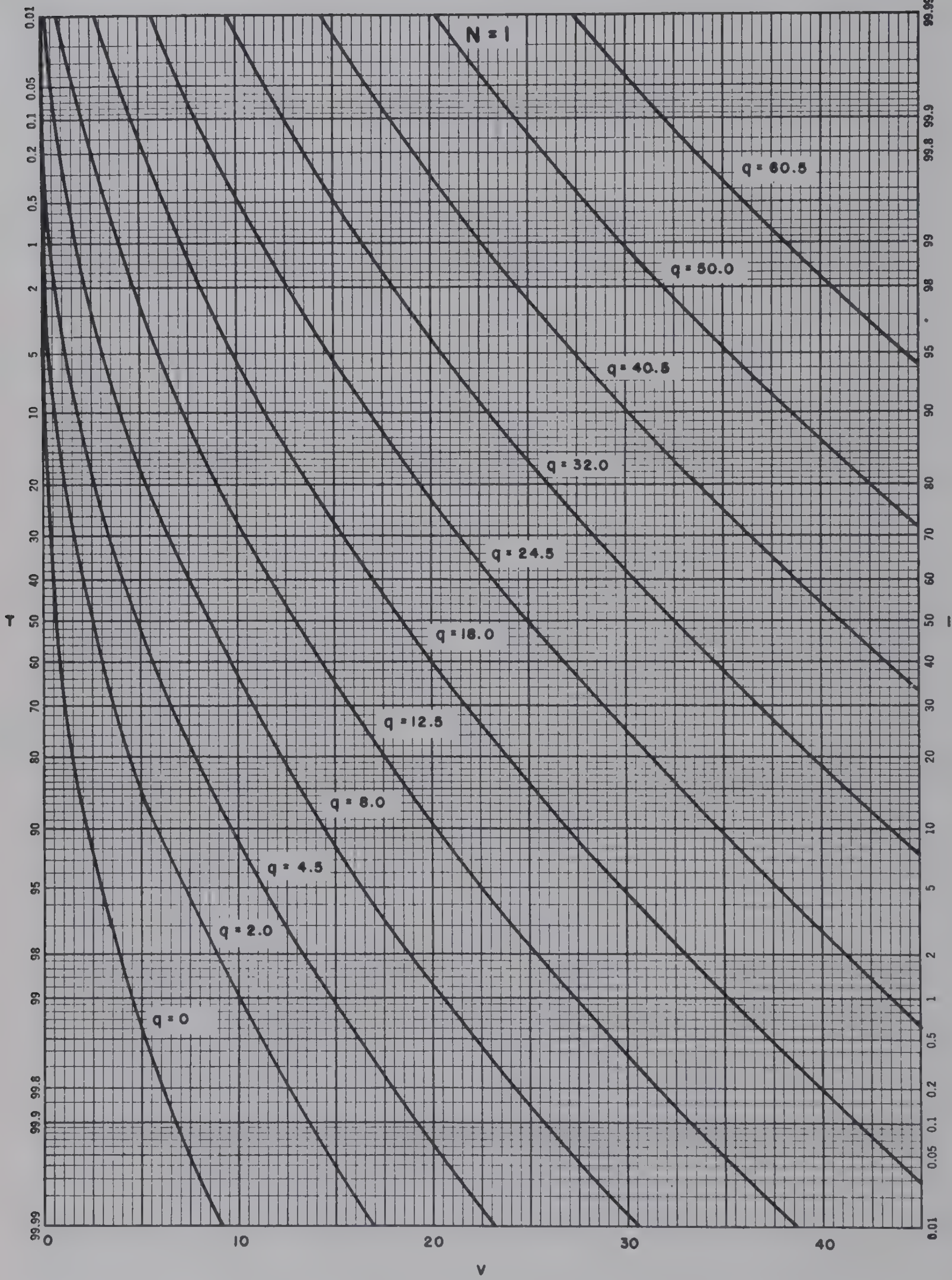
PROBABILITY OF DETECTION WITH NO INTEGRATION

FIG.12



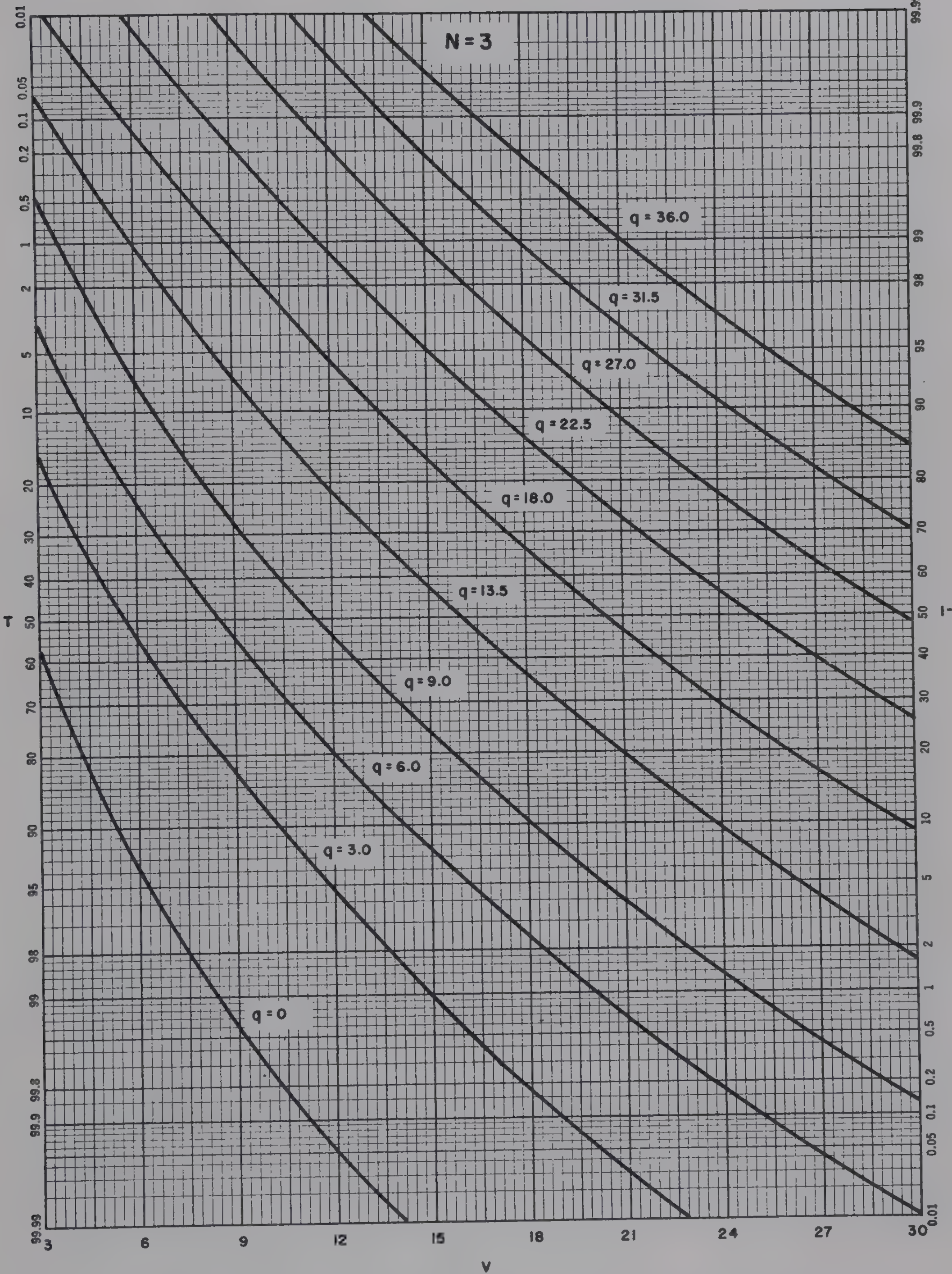
THE INCOMPLETE TORONTO FUNCTION $T_{v\nabla}$ (2N-1, N-1, $\sqrt{q}$)

FIG. 13



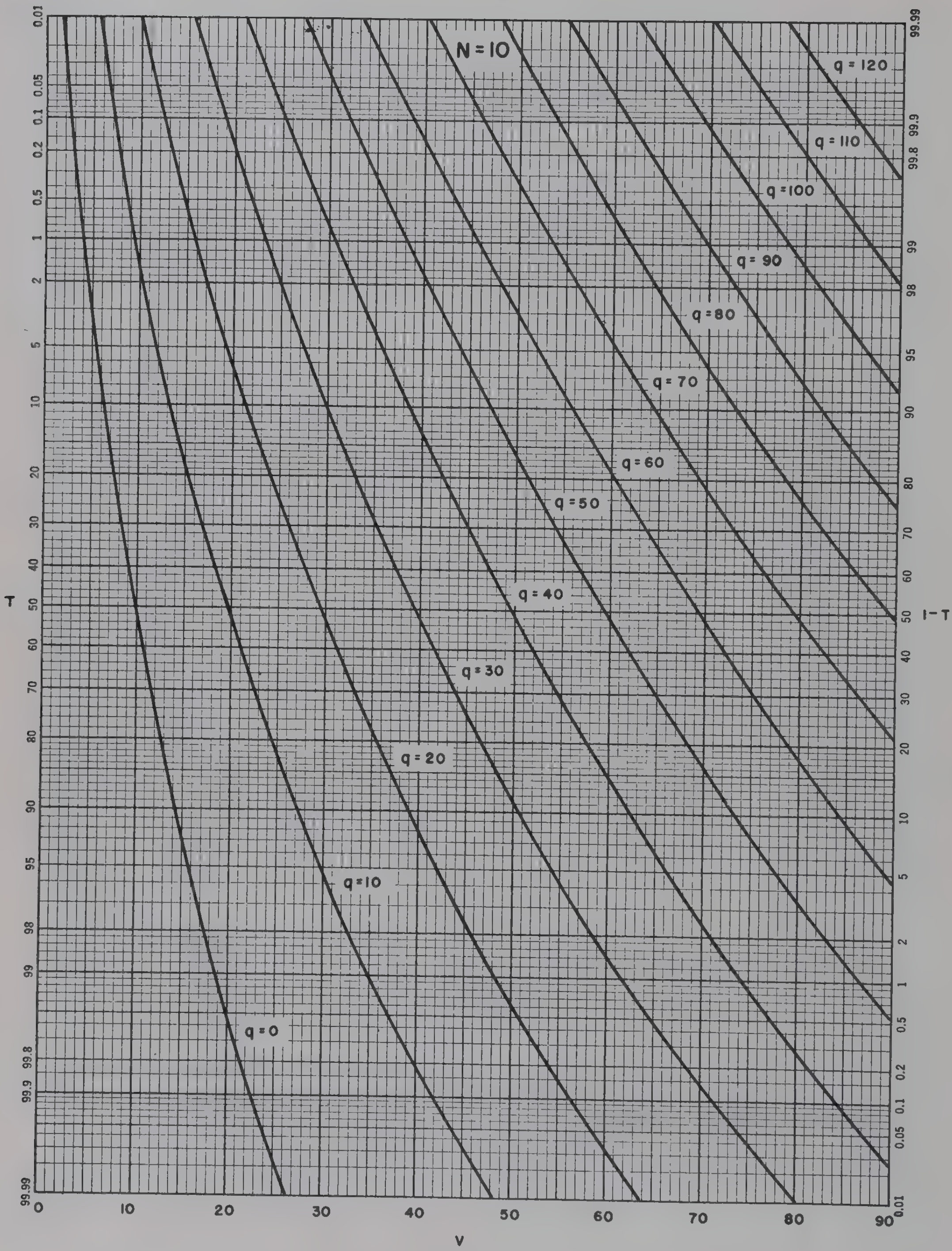
THE INCOMPLETE TORONTO FUNCTION $T_{\sqrt{V}}$ (2N-1, N-1, $\sqrt{q}$)

FIG. 14



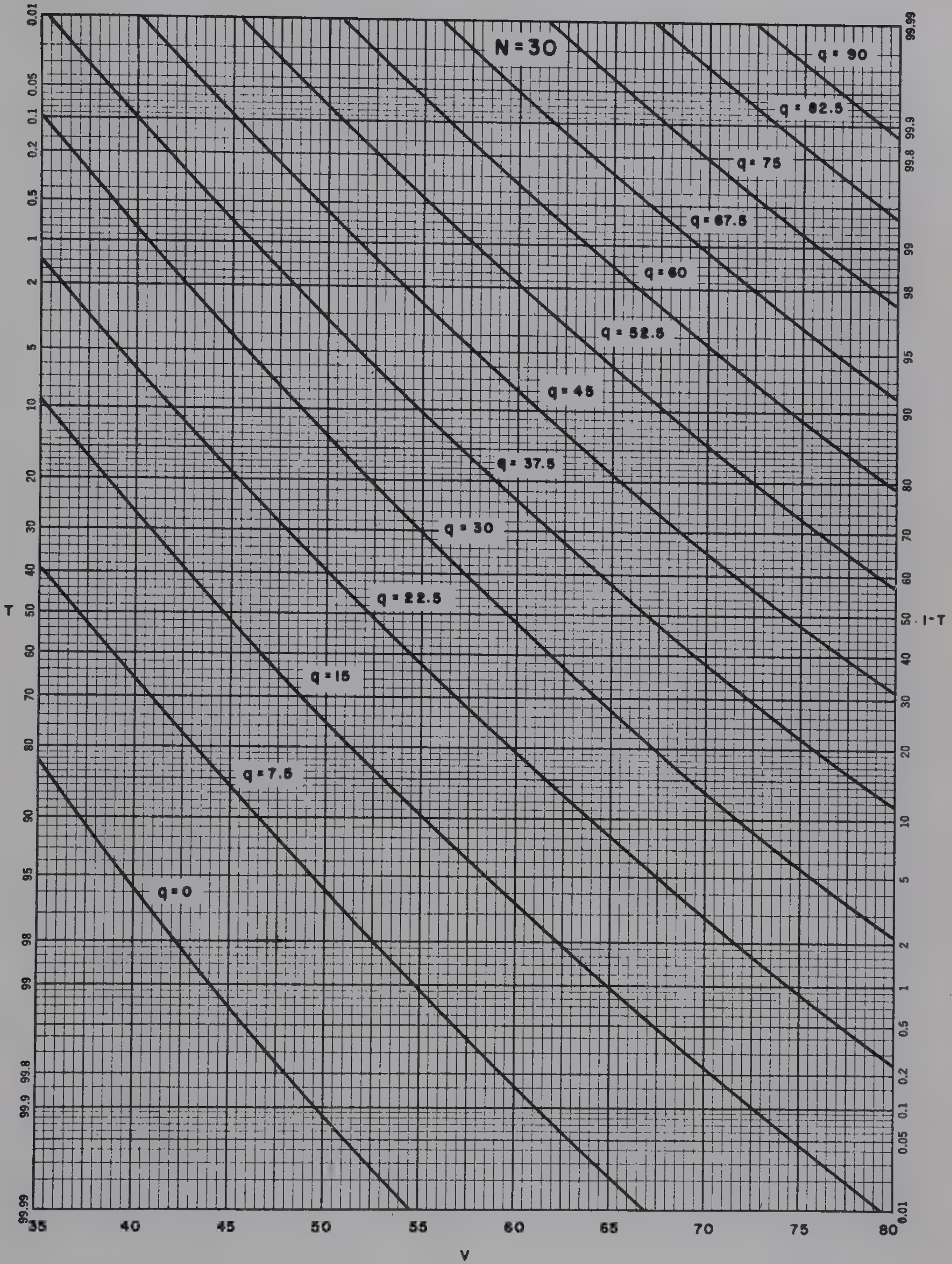
THE INCOMPLETE TORONTO FUNCTION $T_{\gamma V}$ $(2N-1,\ N-1,\ \sqrt{q})$

FIG. 15



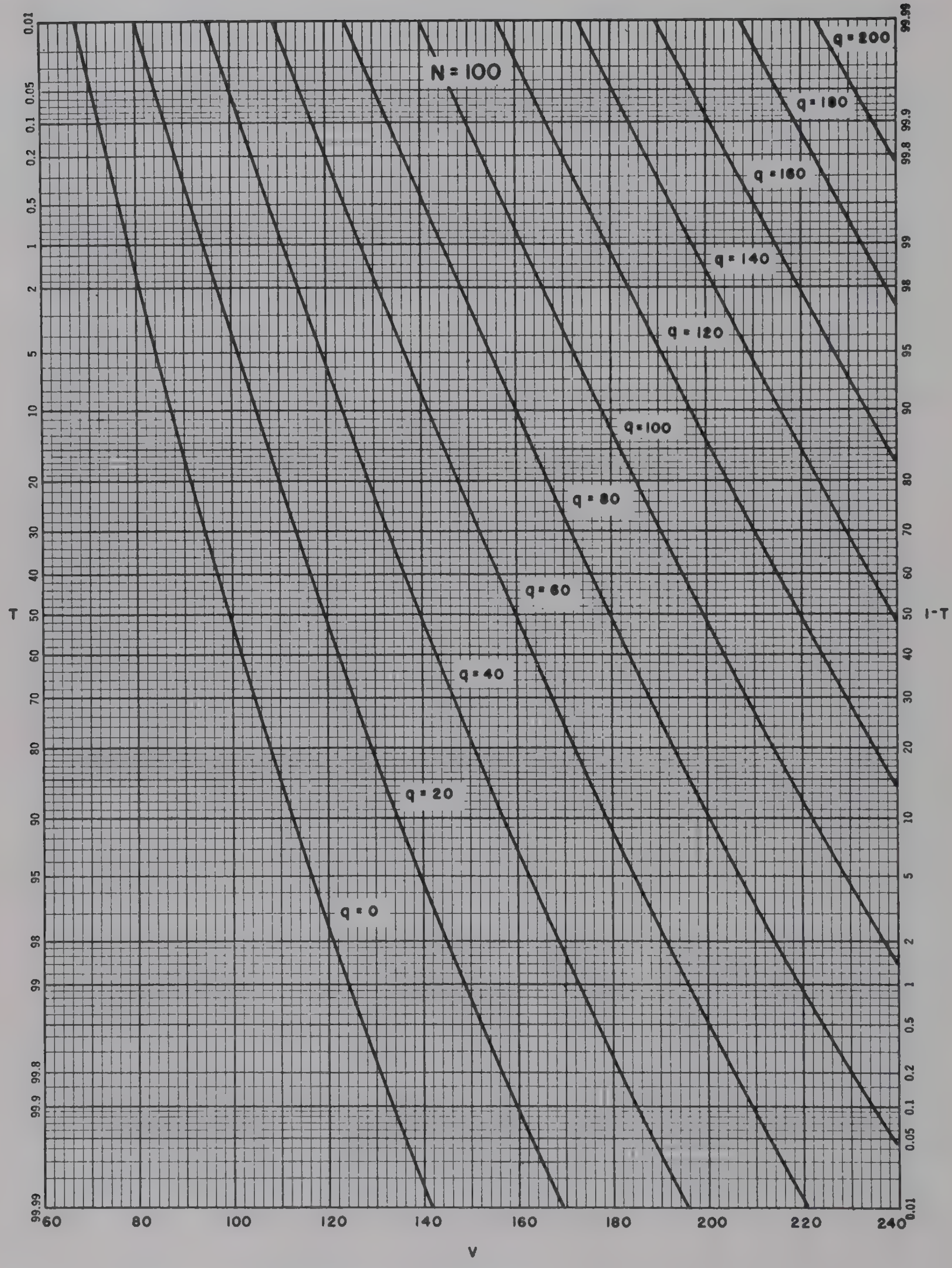
THE INCOMPLETE TORONTO FUNCTION $T_{\gamma V}$ $(2N-1,\ N-1,\ \sqrt{q}\ )$

FIG. 16



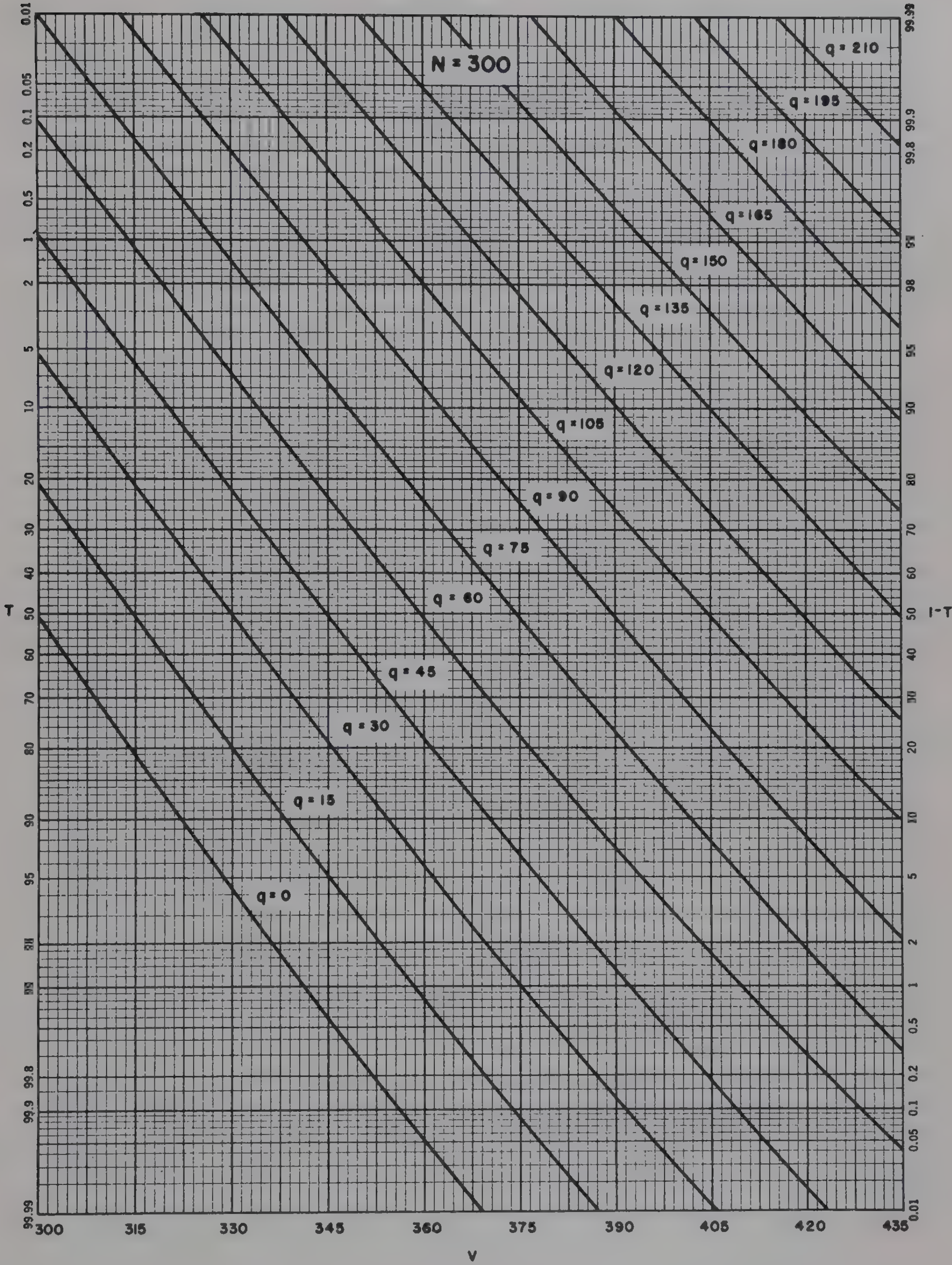
THE INCOMPLETE TORONTO FUNCTION $T_{\sqrt{V}}$ (2N-1, N-1, $\sqrt{q}$)

FIG. 17



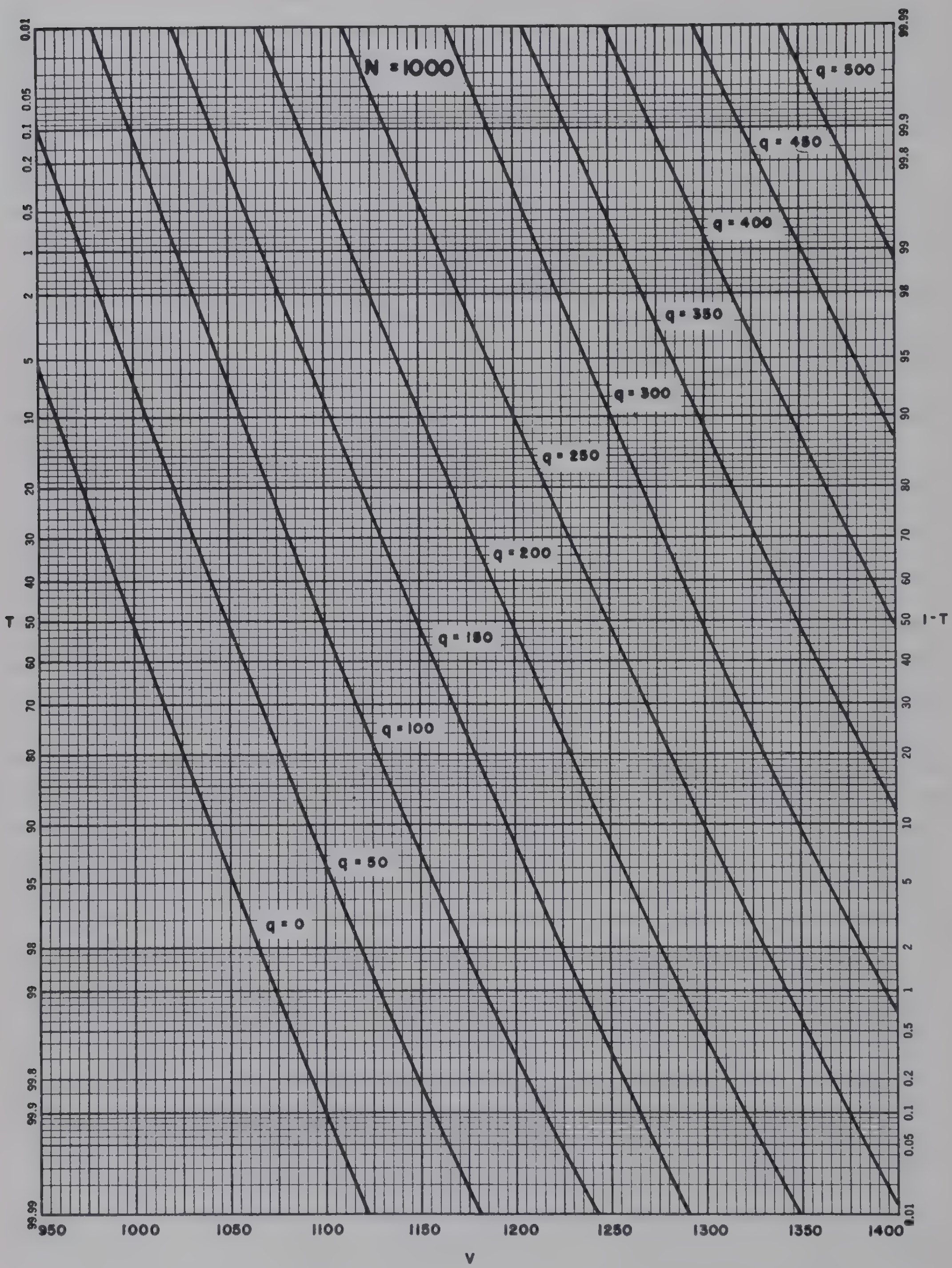
THE INCOMPLETE TORONTO FUNCTION $T_{\sqrt{V}}\ (2N-1,\ N-1,\ \sqrt{q}\ )$

FIG. 18



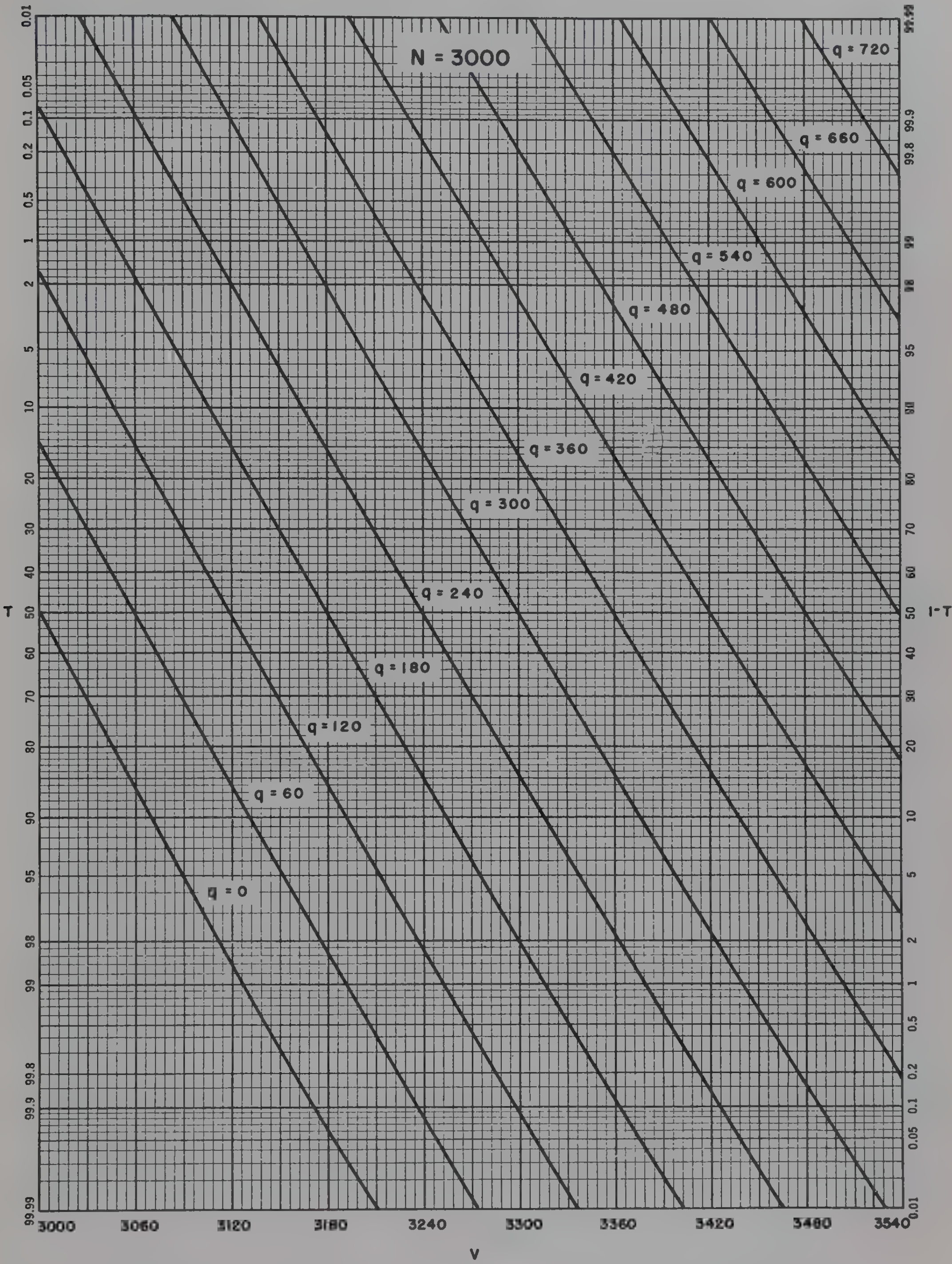
THE INCOMPLETE TORONTO FUNCTION $T_{\sqrt{V}}$ $(2N-1,\ N-1,\ \sqrt{q})$

FIG.19



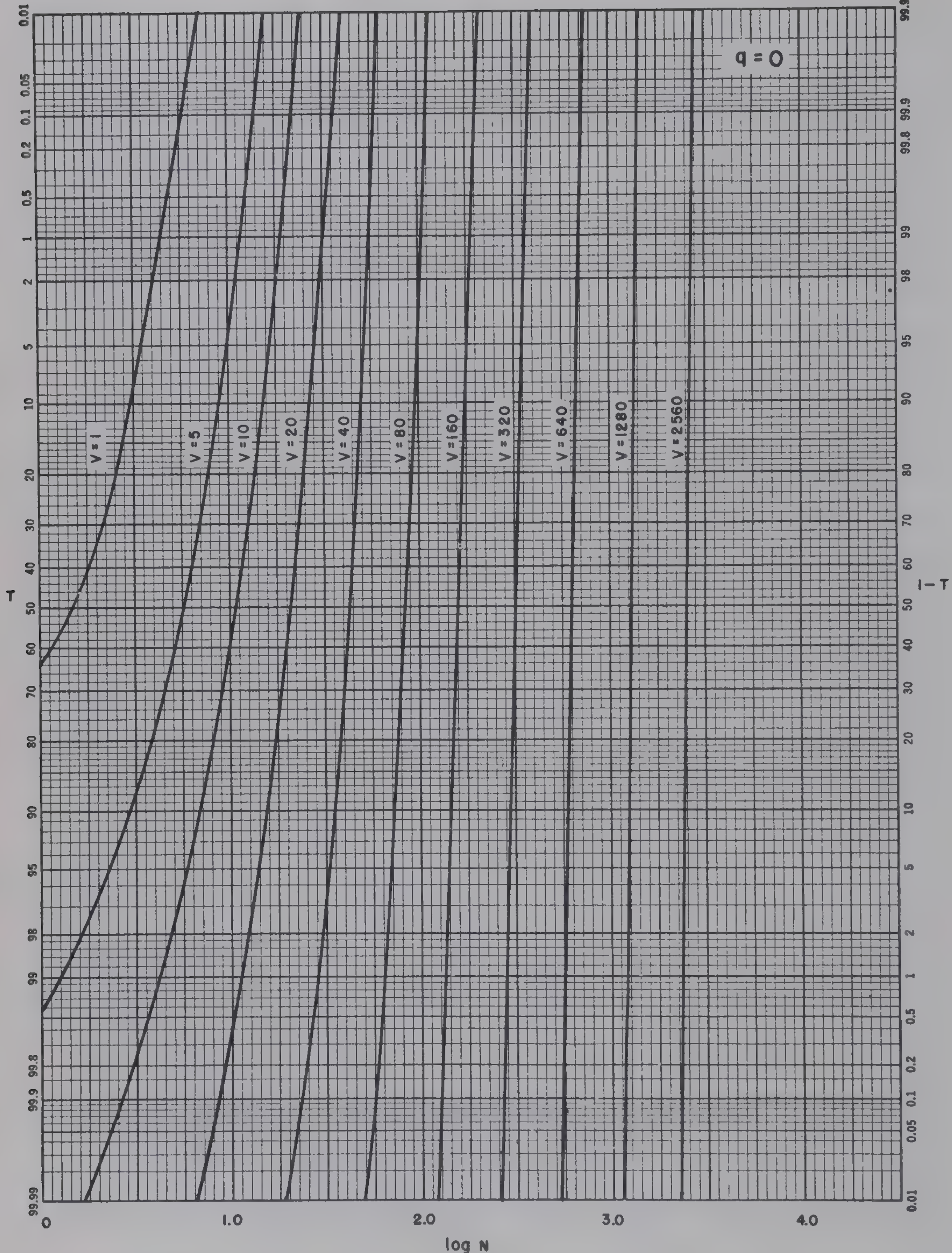
THE INCOMPLETE TORONTO FUNCTION $T_{v\sqrt{V}}$ (2N-1, N-1, $\sqrt{q}$)

FIG. 20



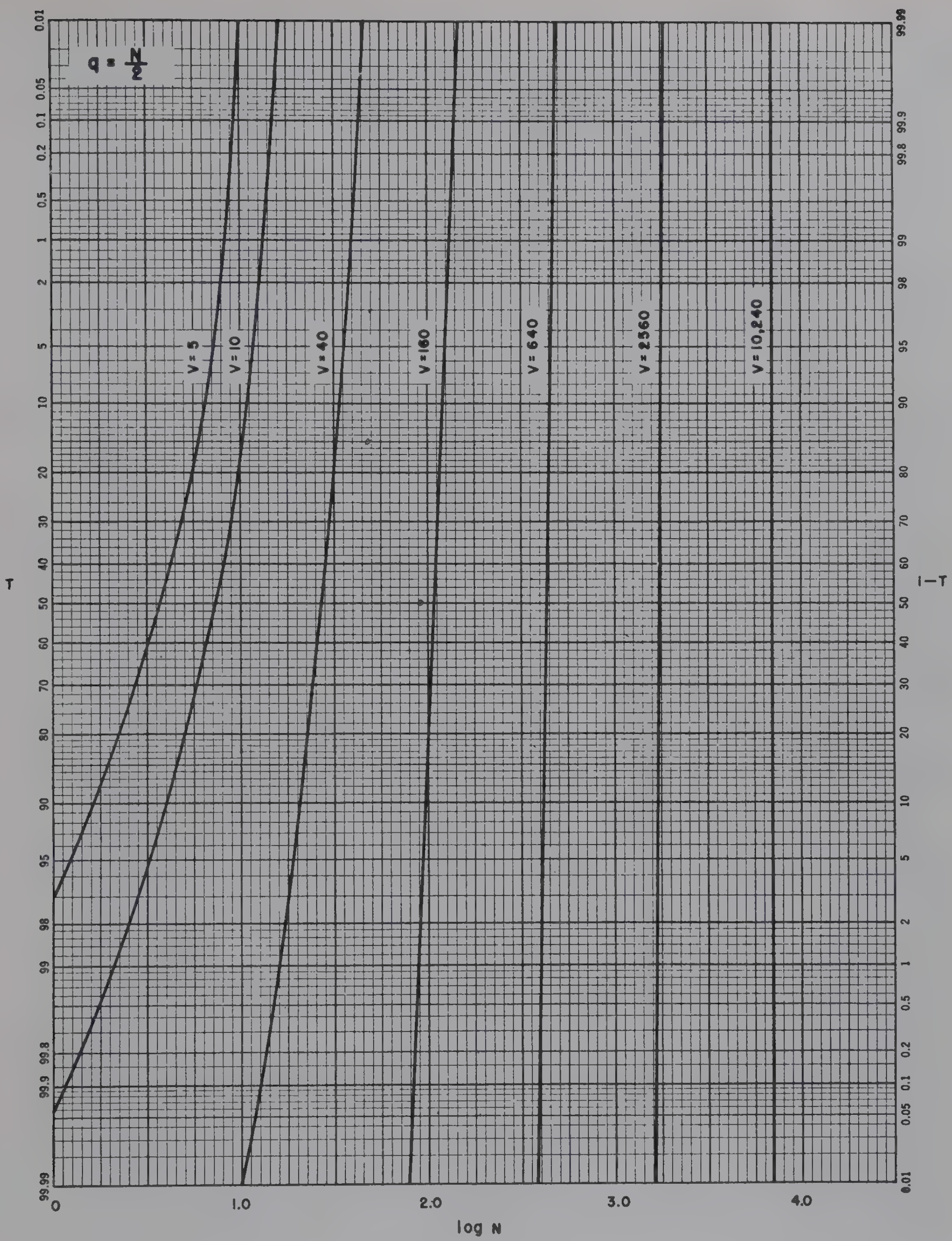
THE INCOMPLETE TORONTO FUNCTION $T_{\sqrt{V}}$ (2N-1, N-1, $\sqrt{q}$)

FIG. 21



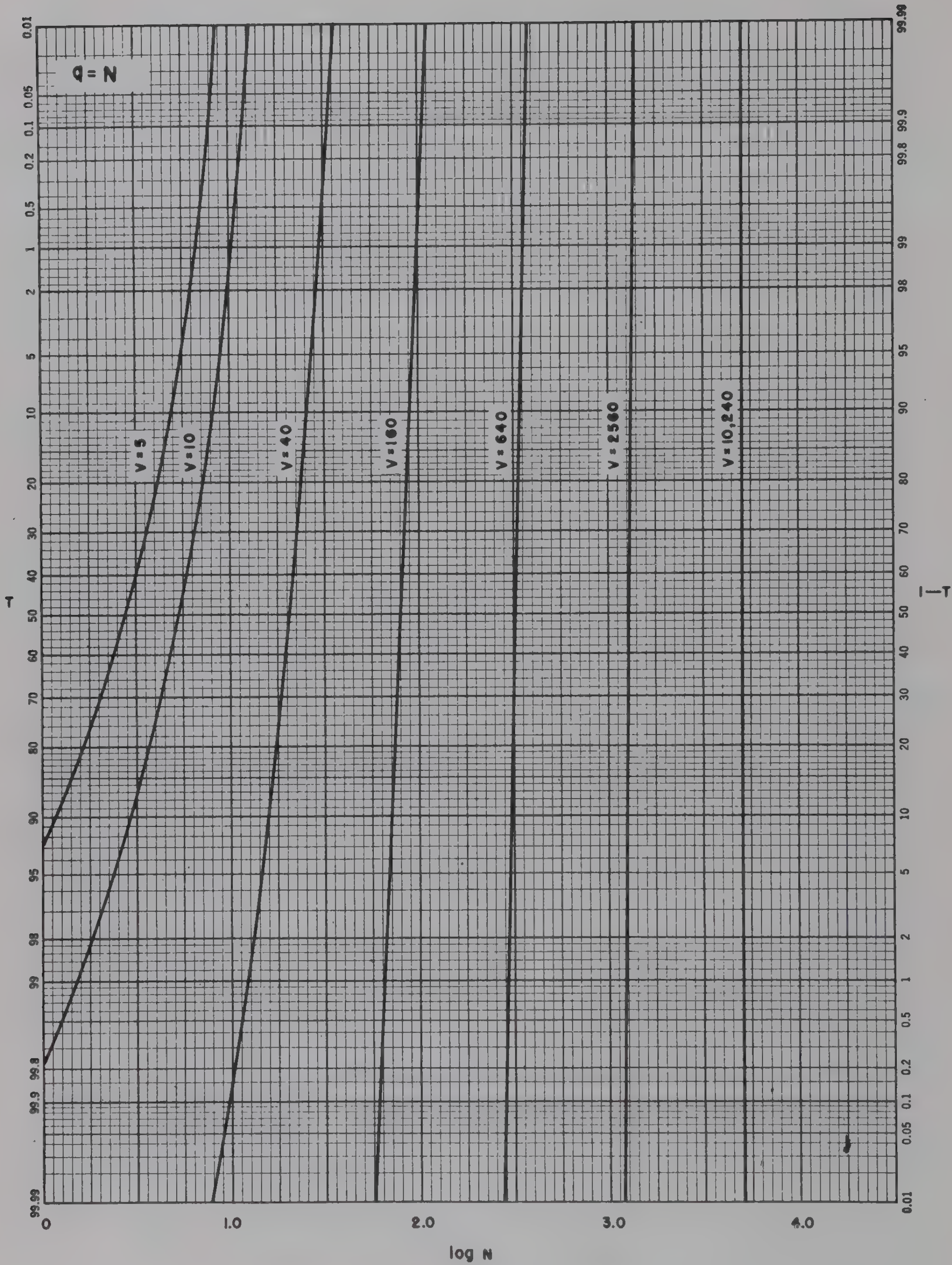
THE INCOMPLETE TORONTO FUNCTION $T_{\sqrt{V}}$ (2N-1, N-1, $\sqrt{q}$)

FIG. 22



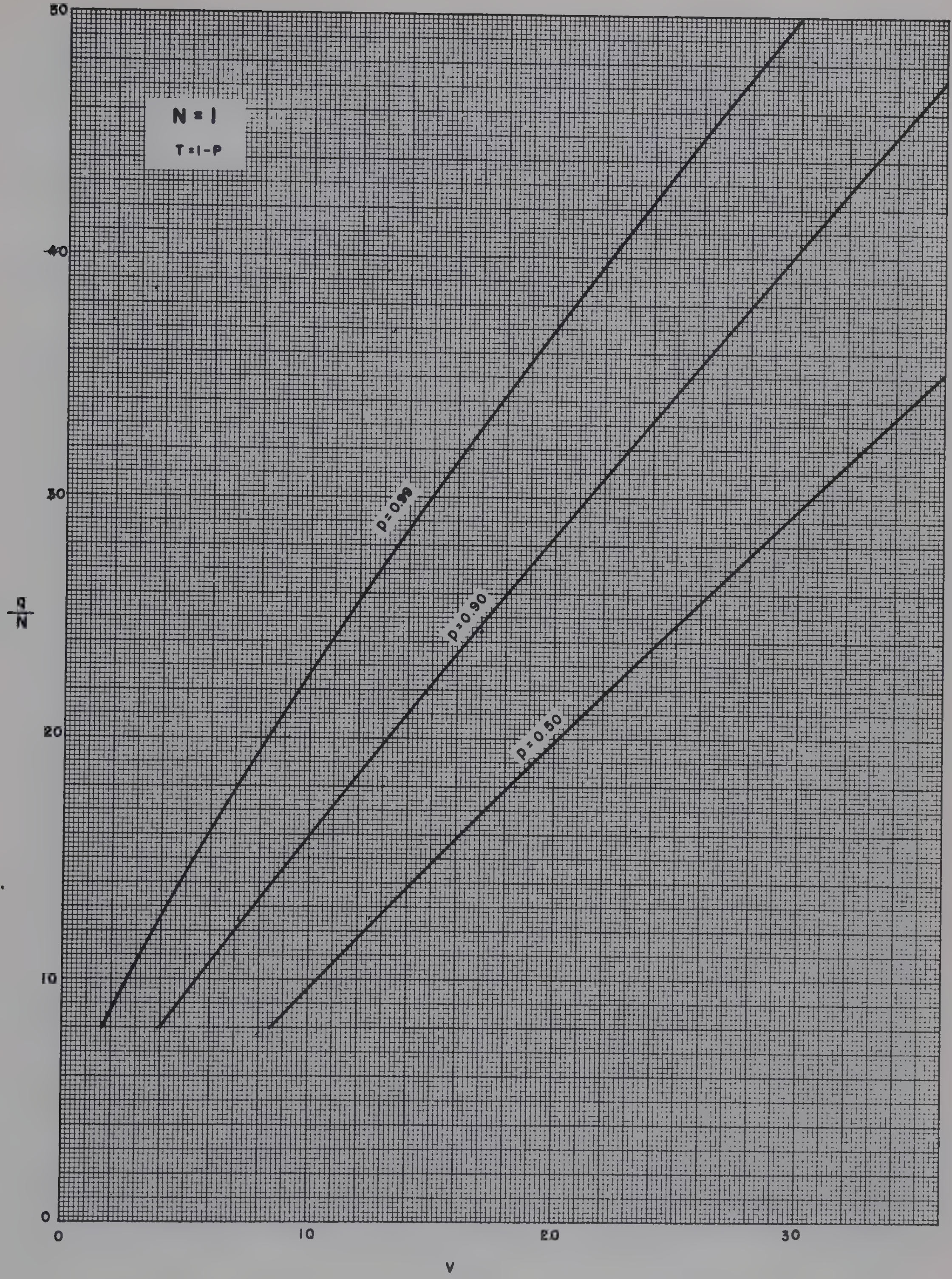
THE INCOMPLETE TORONTO FUNCTION $T_{\sqrt{V}}$ (2N−1, N−1, $\sqrt{q}$)

FIG. 23

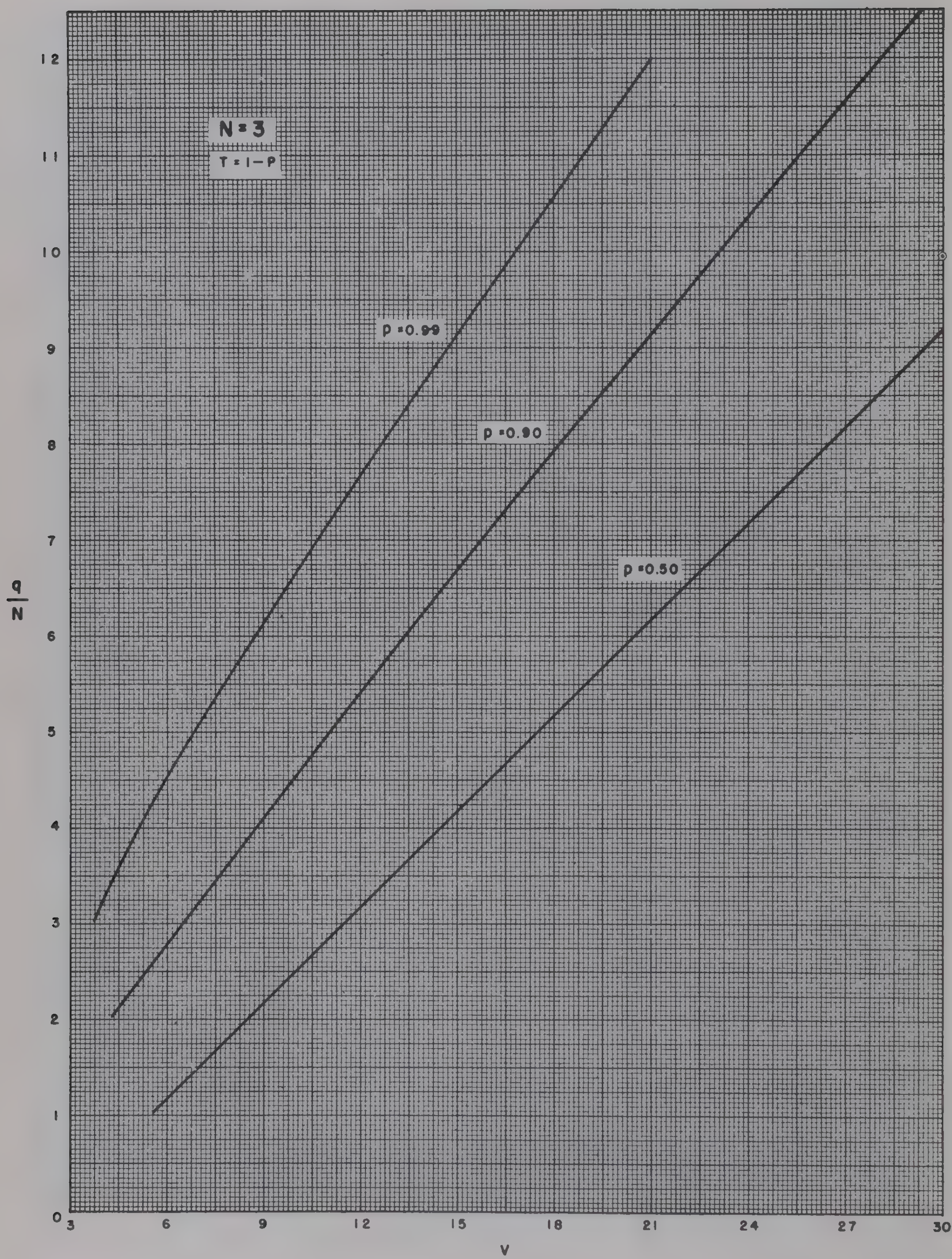


THE INCOMPLETE TORONTO FUNCTION $T_{\sqrt{V}}\ (2N-1,\ N-1,\ \sqrt{q}\ )$

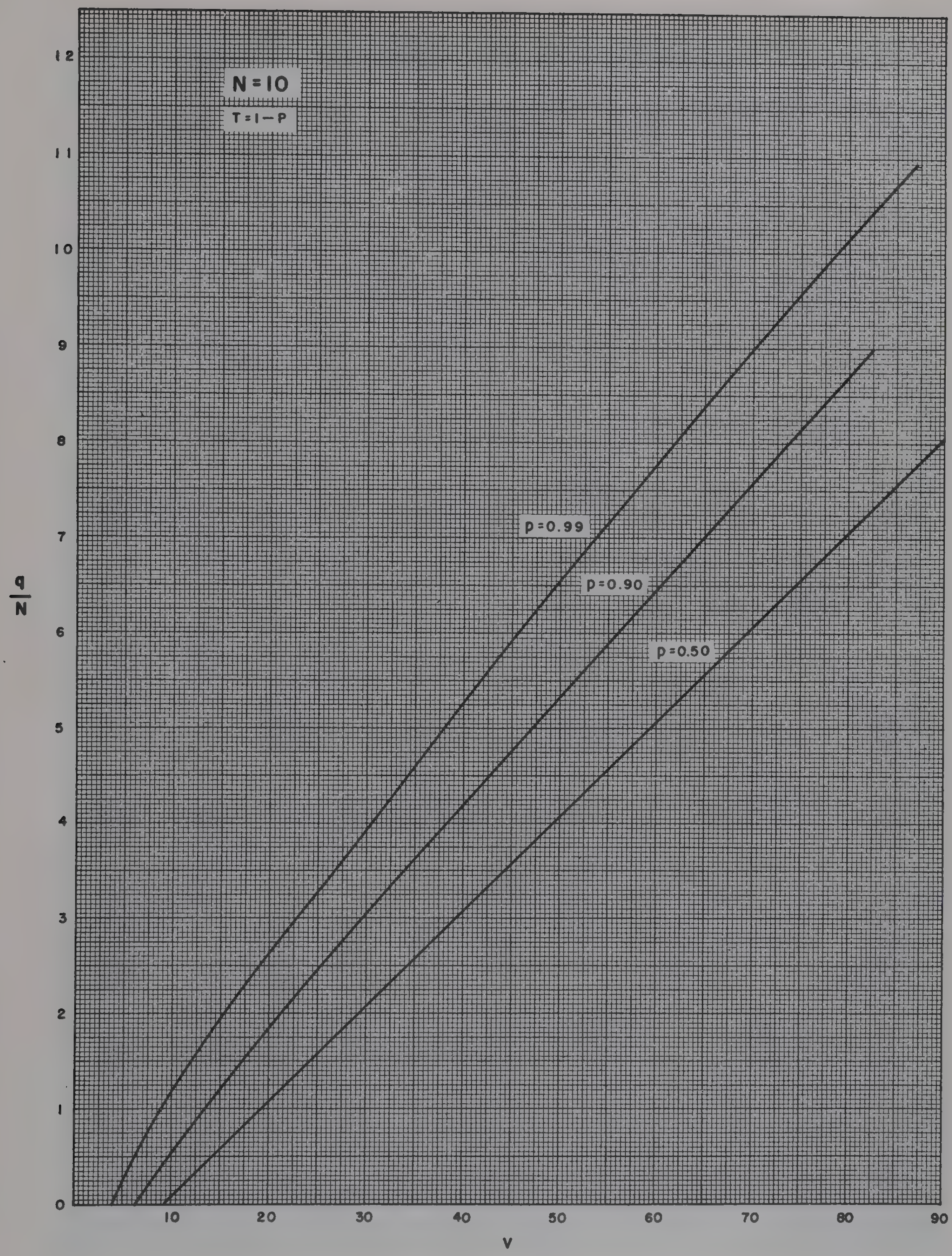
FIG. 24



THE INCOMPLETE TORONTO FUNCTION $T_{\sqrt{V}}$ (2N−1, N−1, $\sqrt{q}$)
FIG. 25

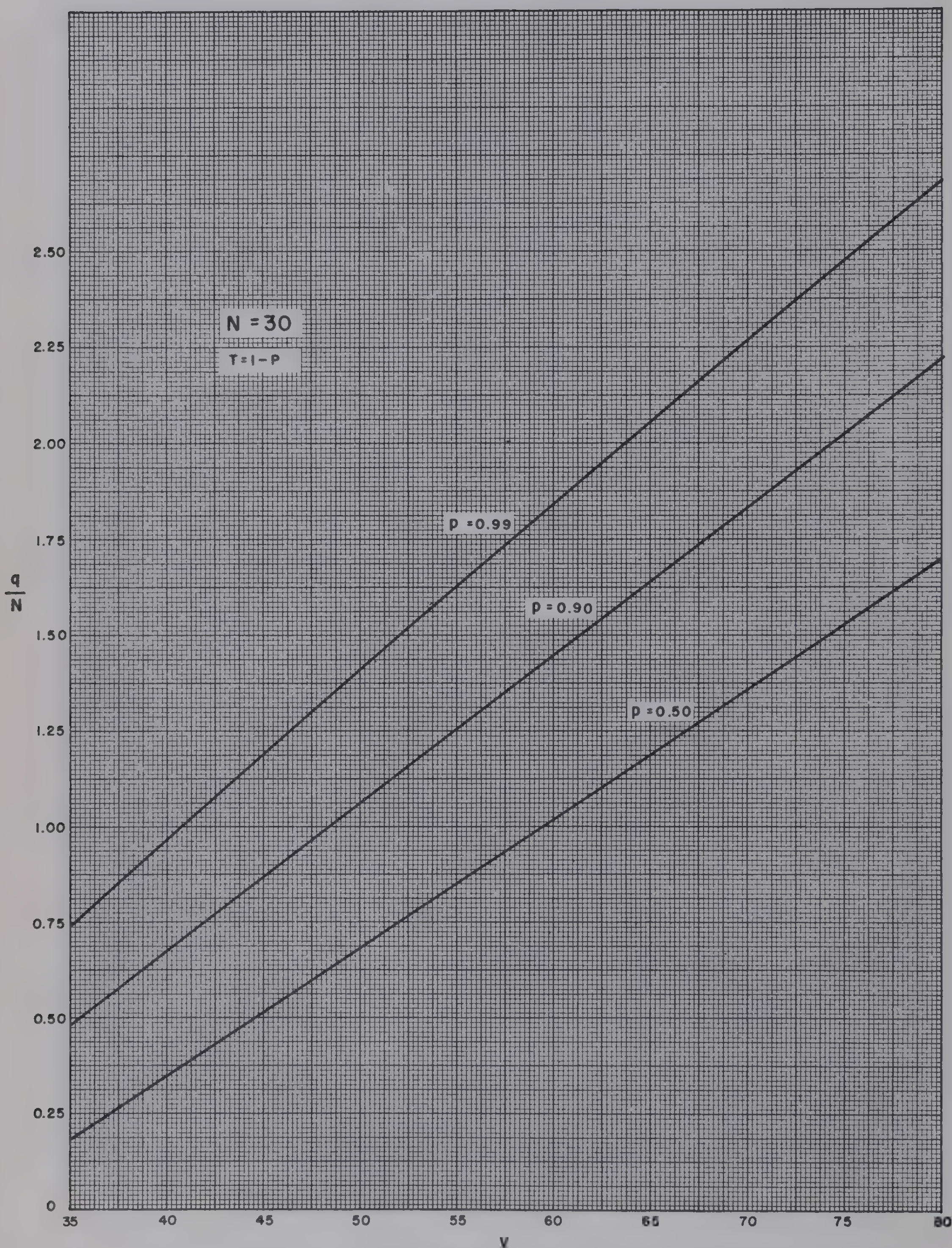


THE INCOMPLETE TORONTO FUNCTION $T_{\sqrt{v}}\ (2N-1,\ N-1,\ \sqrt{q}\ )$
FIG. 26

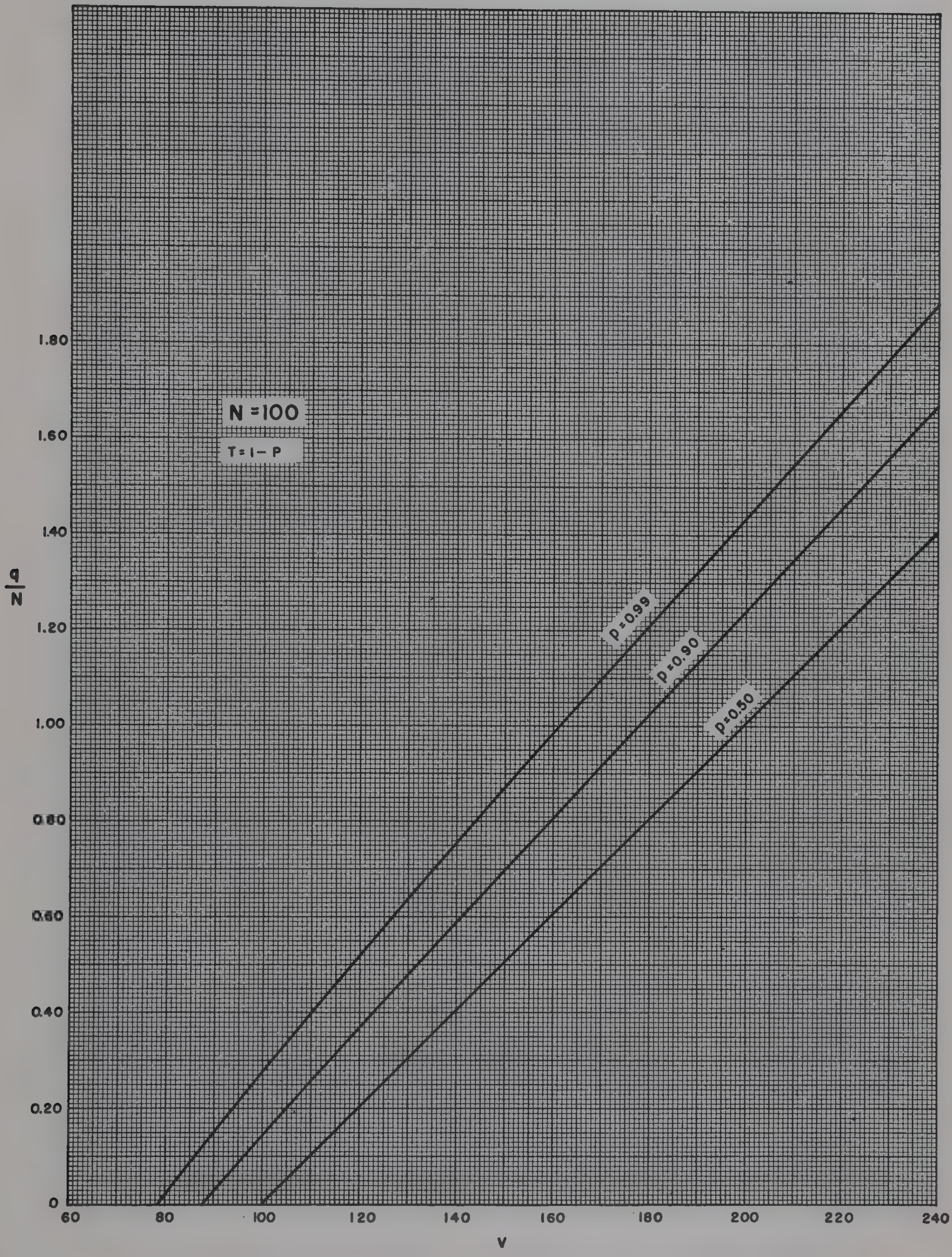


THE INCOMPLETE TORONTO FUNCTION $T_{\sqrt{V}}\ (2N-1,\ N-1,\ \sqrt{q}\ )$

FIG. 27

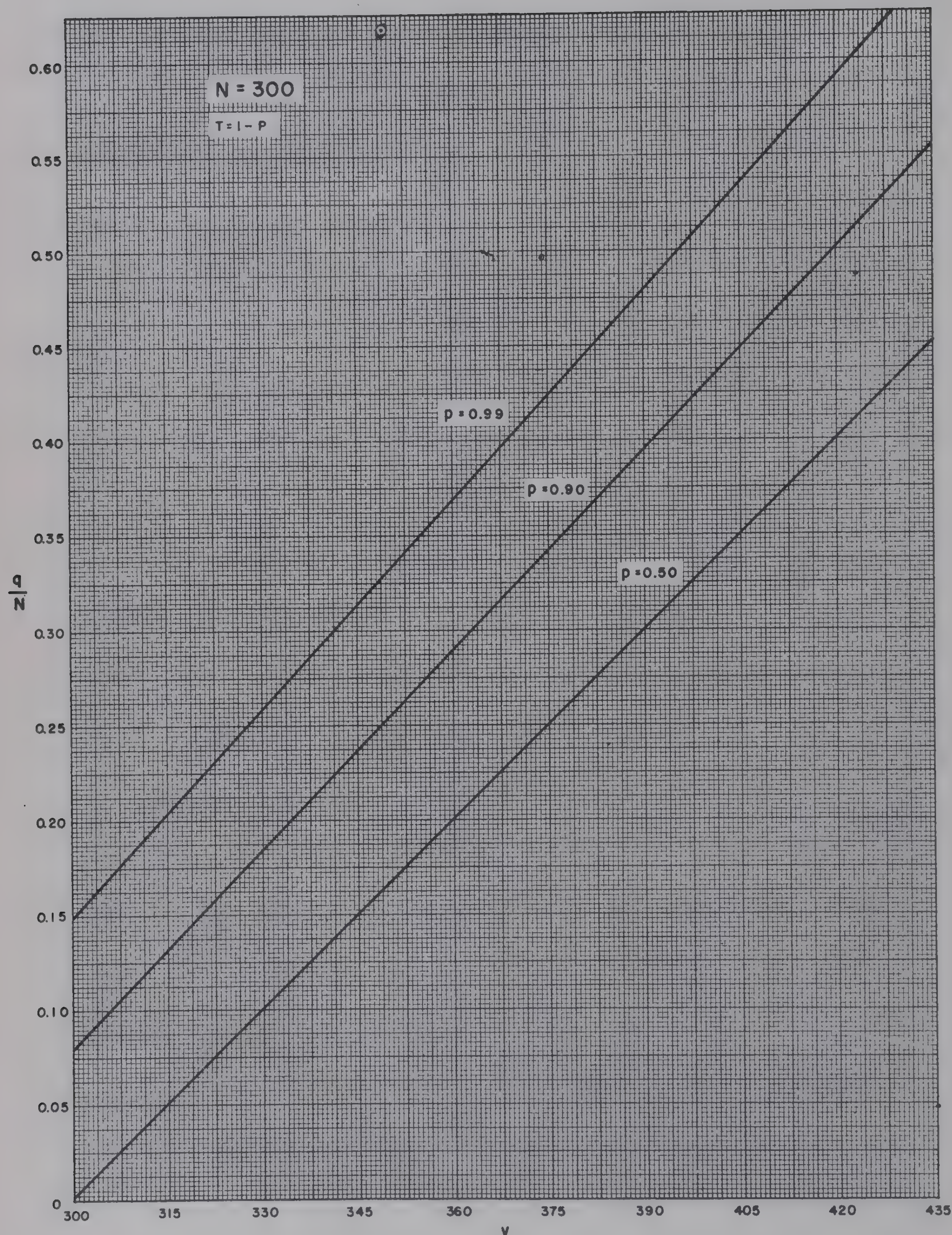


THE INCOMPLETE TORONTO FUNCTION $T_{\sqrt{V}}\ (2N-1,\ N-1,\ \sqrt{q})$
FIG. 28

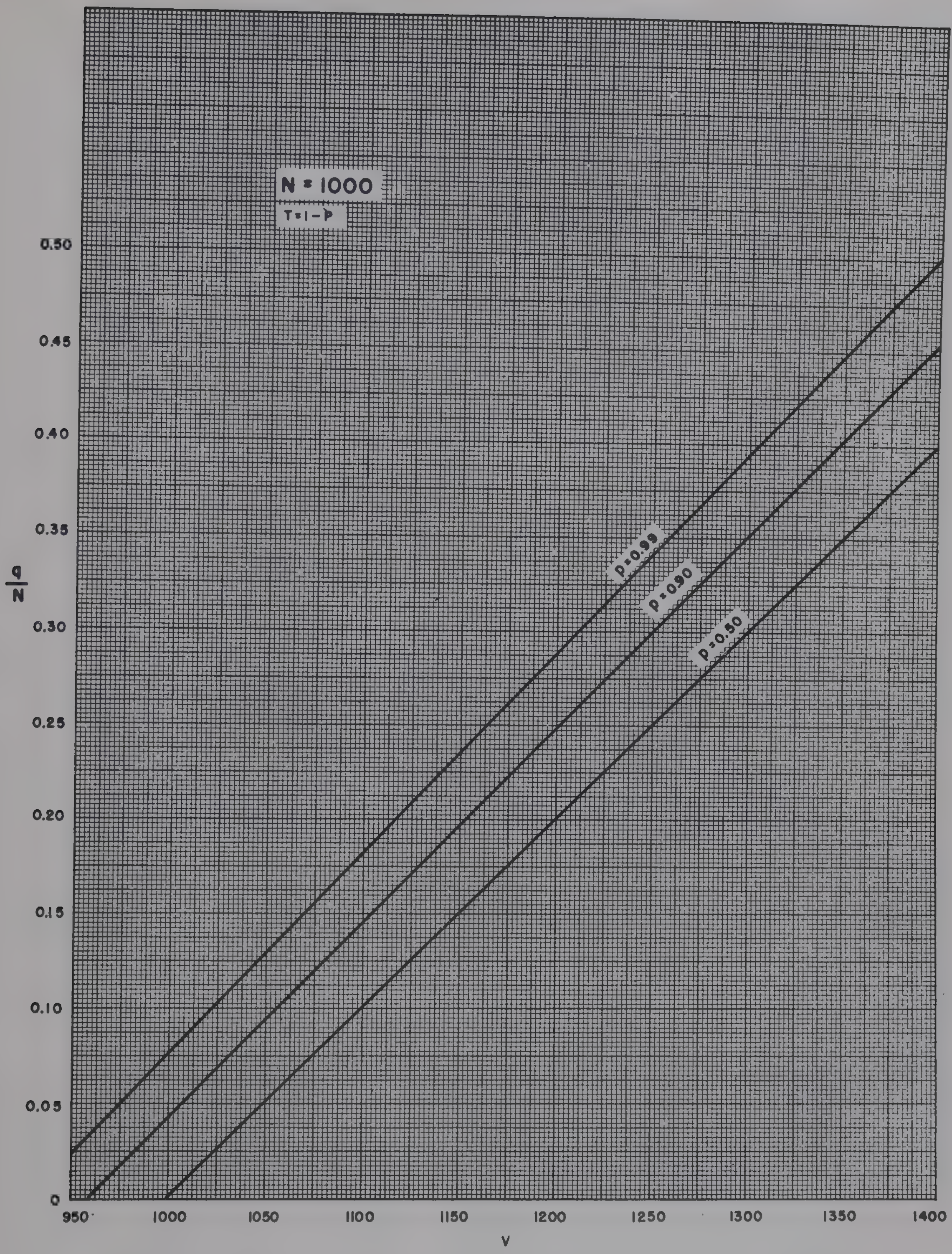


THE INCOMPLETE TORONTO FUNCTION $T_{\sqrt{v}}$ $(2N-1,\ N-1,\ \sqrt{q})$

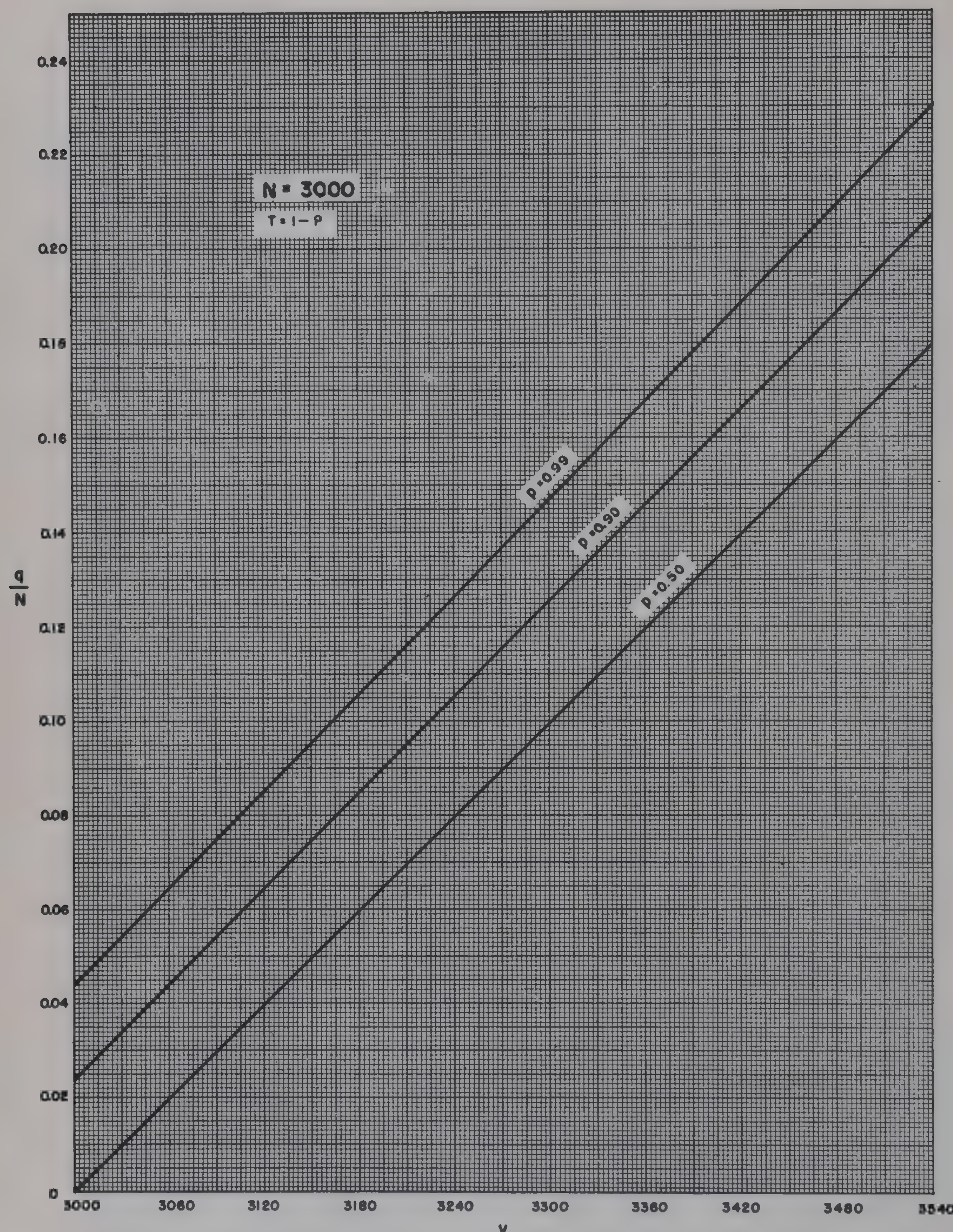
FIG. 29



THE INCOMPLETE TORONTO FUNCTION $T_{\sqrt{v}}$ $(2N-1,\ N-1,\ \sqrt{q})$
FIG.30

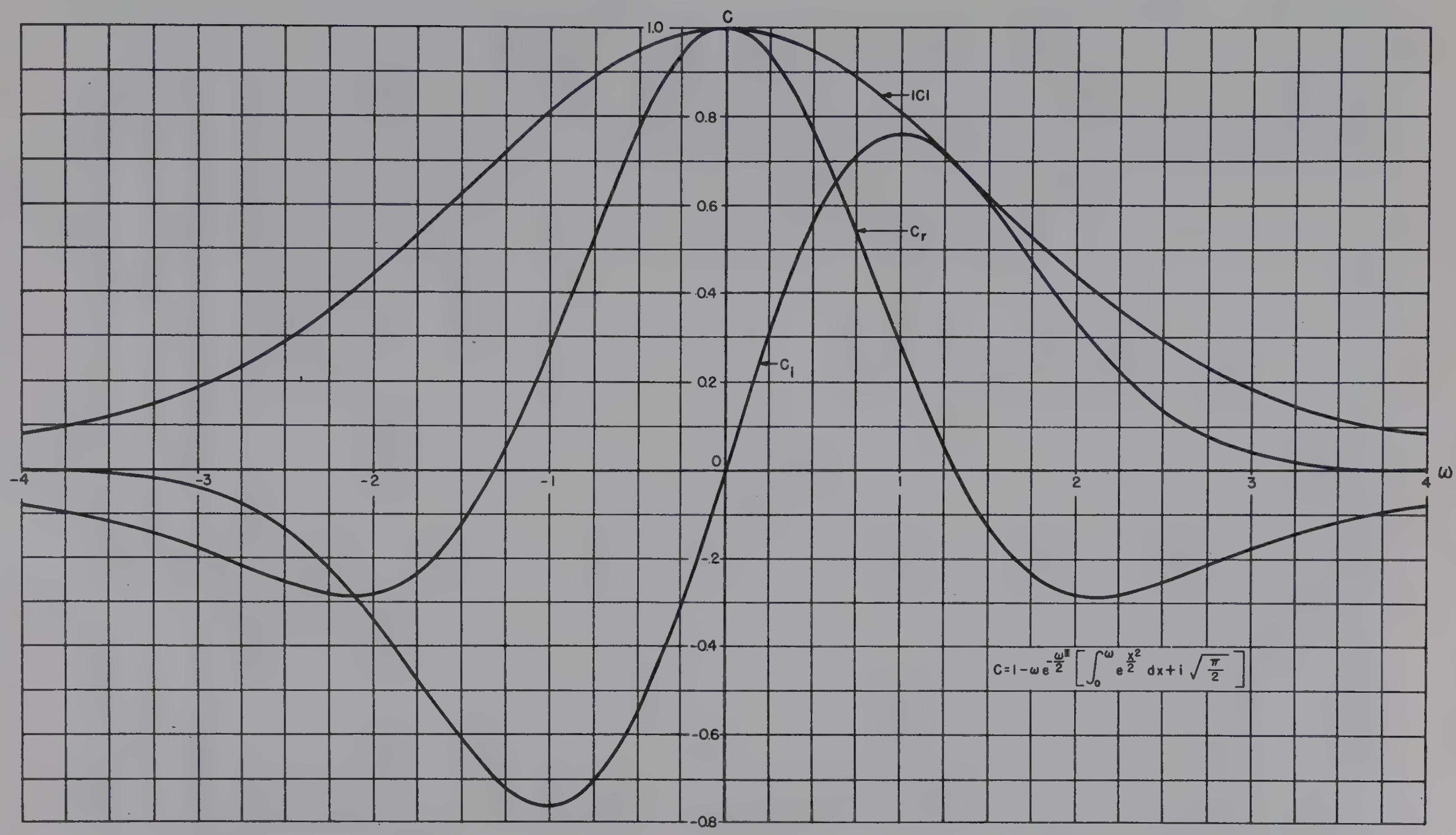


THE INCOMPLETE TORONTO FUNCTION $T_{\sqrt{V}}$ $(2N-1, N-1, \sqrt{q})$
FIG. 31



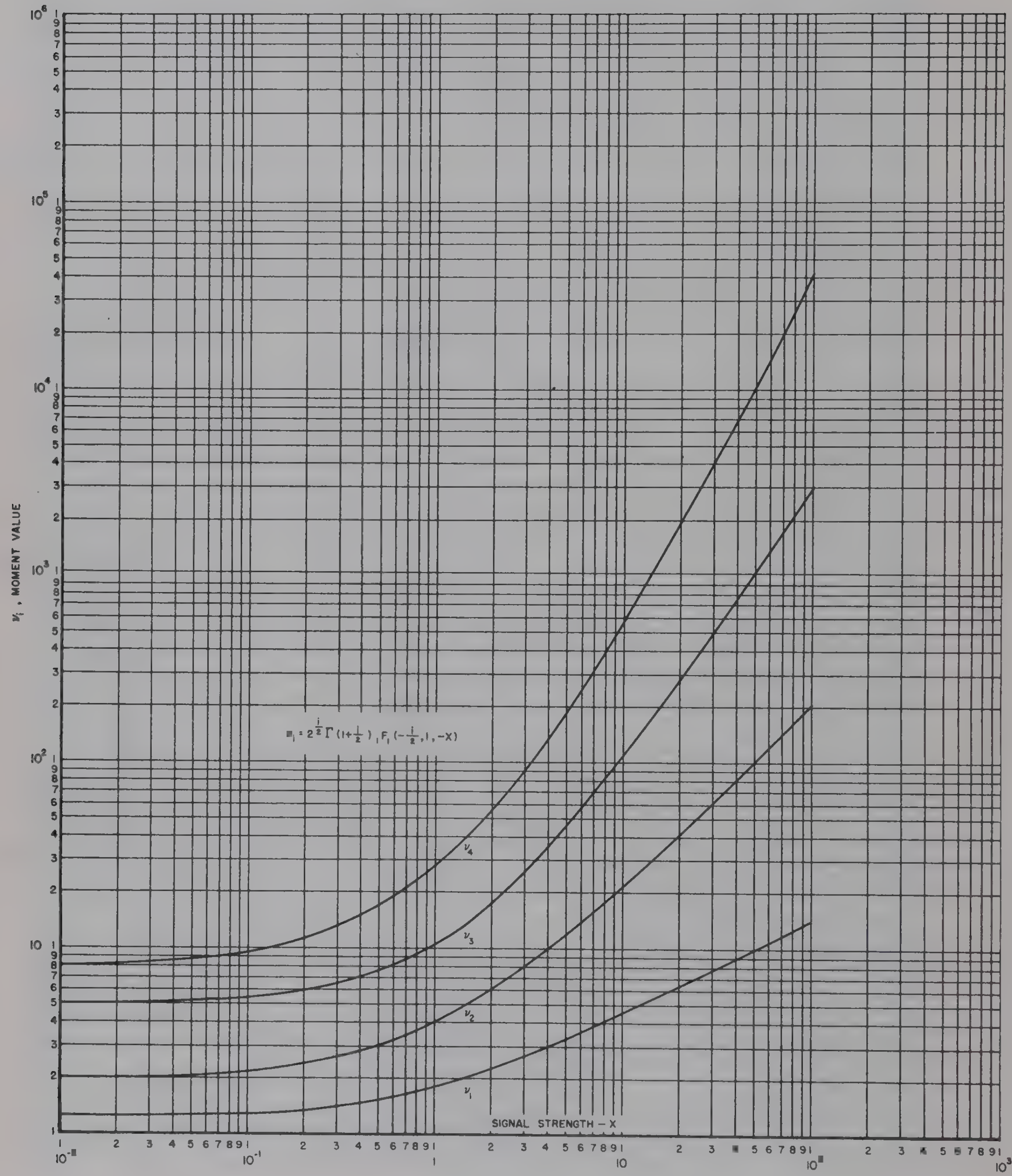
THE INCOMPLETE TORONTO FUNCTION $T_{\sqrt{V}}$ $(2N-1,\ N-1,\ \sqrt{q})$

FIG. 32

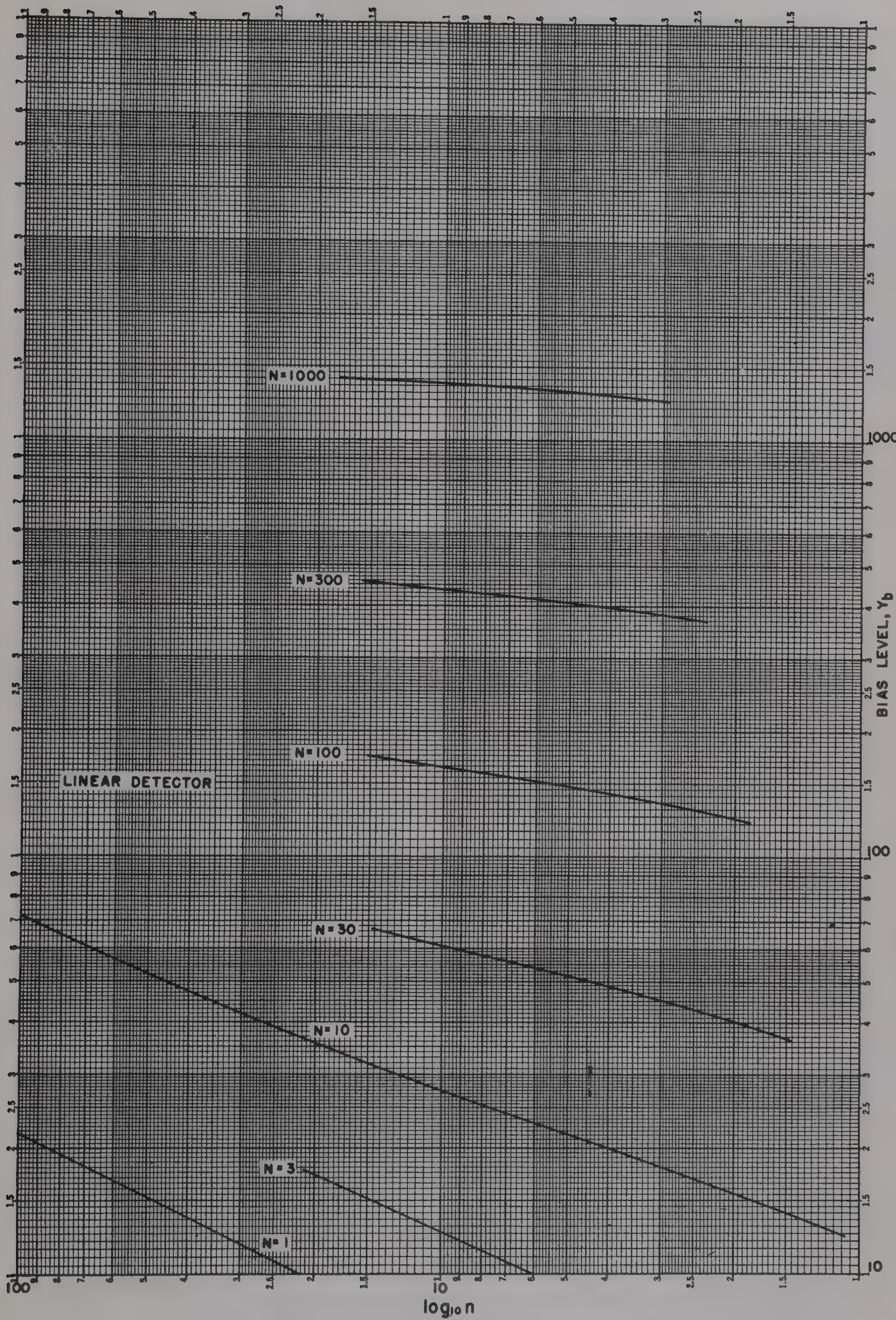


CHARACTERISTIC FUNCTION FOR ENVELOPE OF NOISE

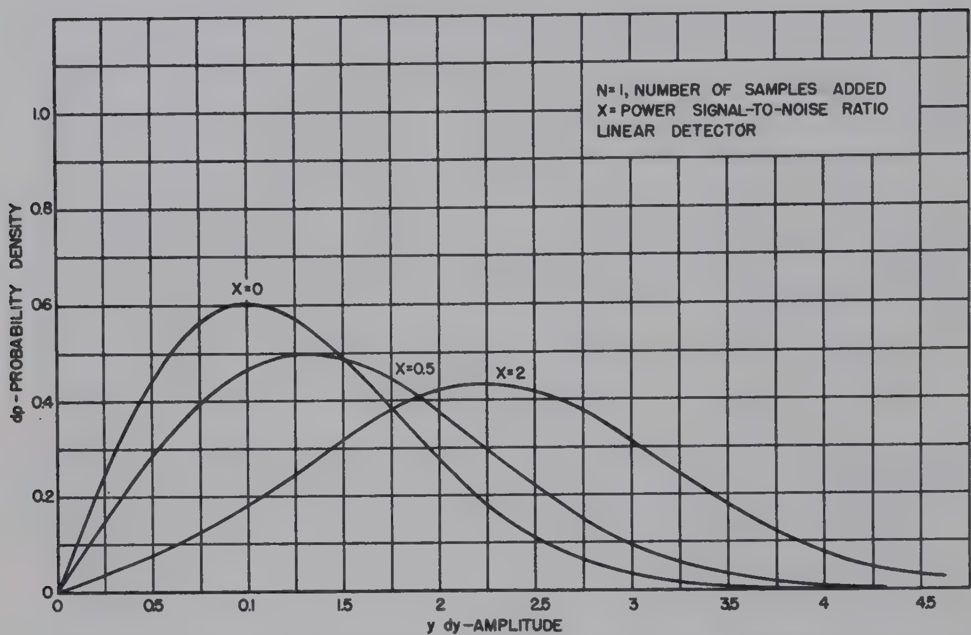
FIG. 33



MOMENTS FOR SIGNAL PLUS NOISE, LINEAR DETECTOR, N=1
FIG. 34

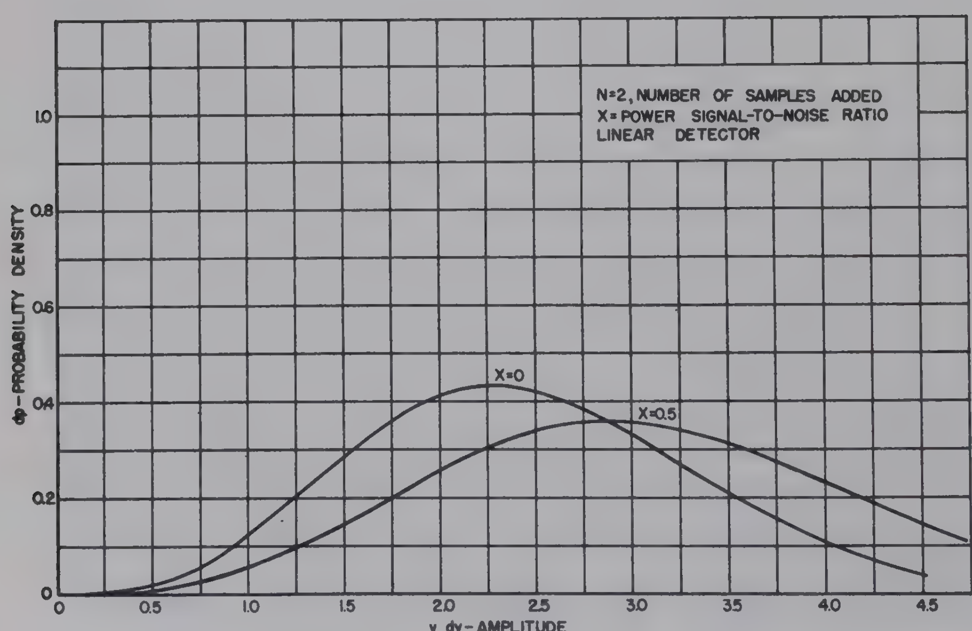


BIAS LEVEL AS A FUNCTION OF NUMBER OF PULSES INTEGRATED AND FALSE ALARM NUMBER
FIG. 35



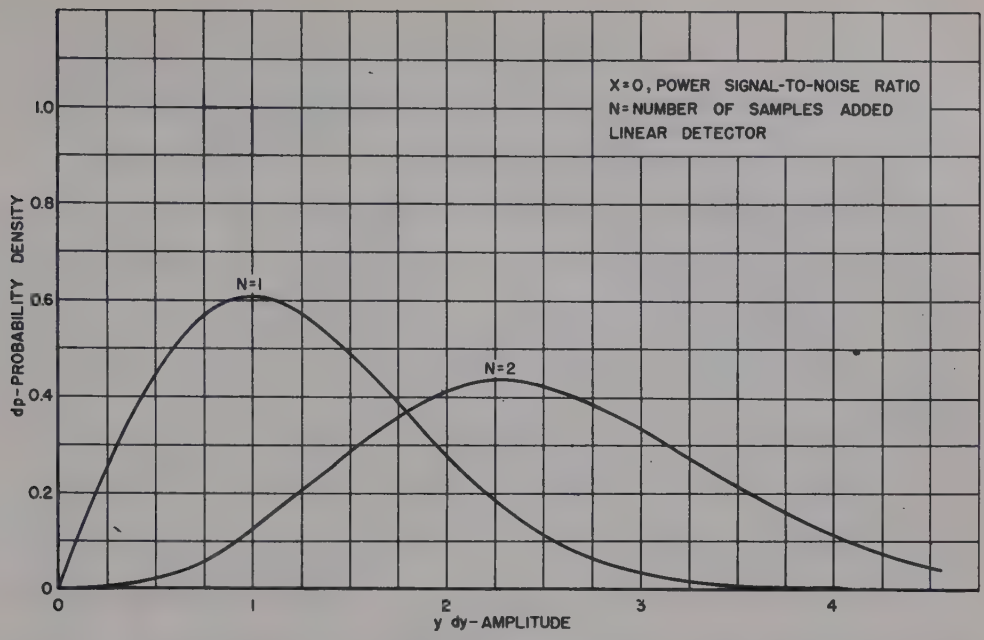
PROBABILITY DENSITY FUNCTIONS FOR ENVELOPE OF SINE WAVE PLUS NOISE

FIG. 36



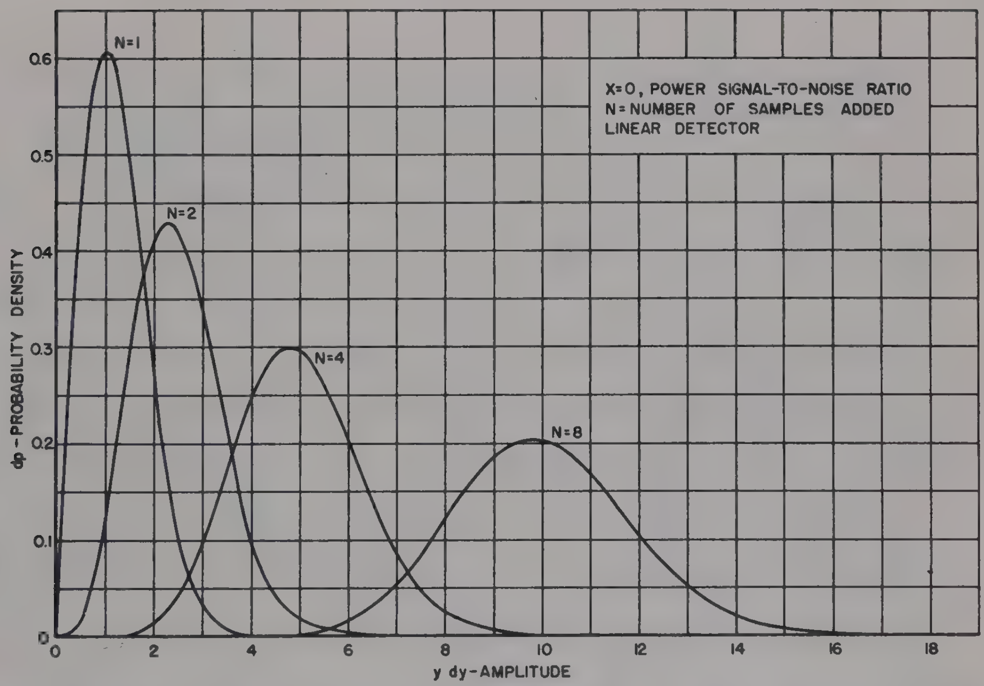
PROBABILITY DENSITY FUNCTIONS FOR ENVELOPE OF SINE WAVE PLUS NOISE

FIG. 37



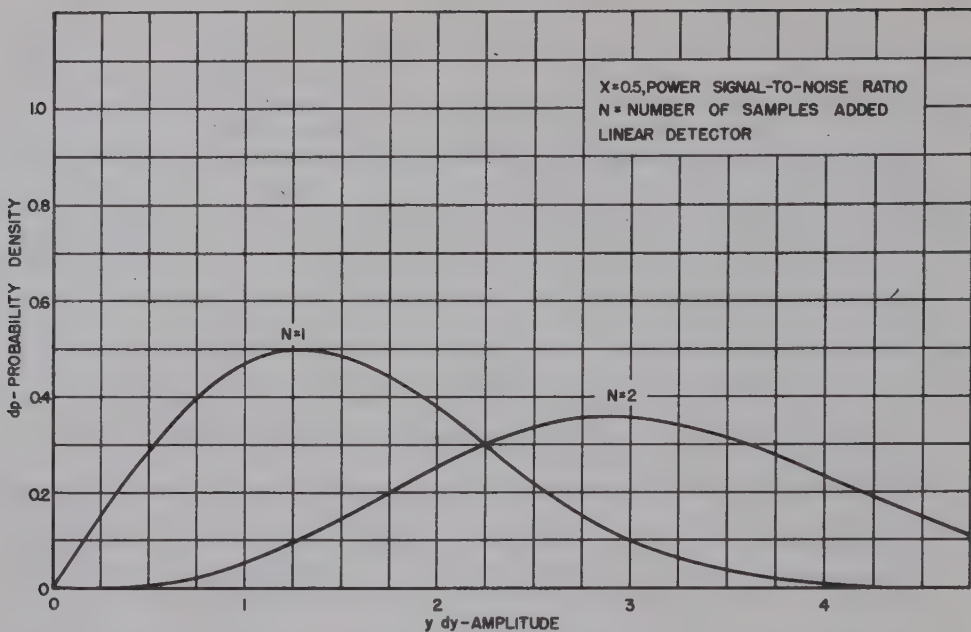
PROBABILITY DENSITY FUNCTIONS FOR ENVELOPE OF SINE WAVE PLUS NOISE

FIG. 38



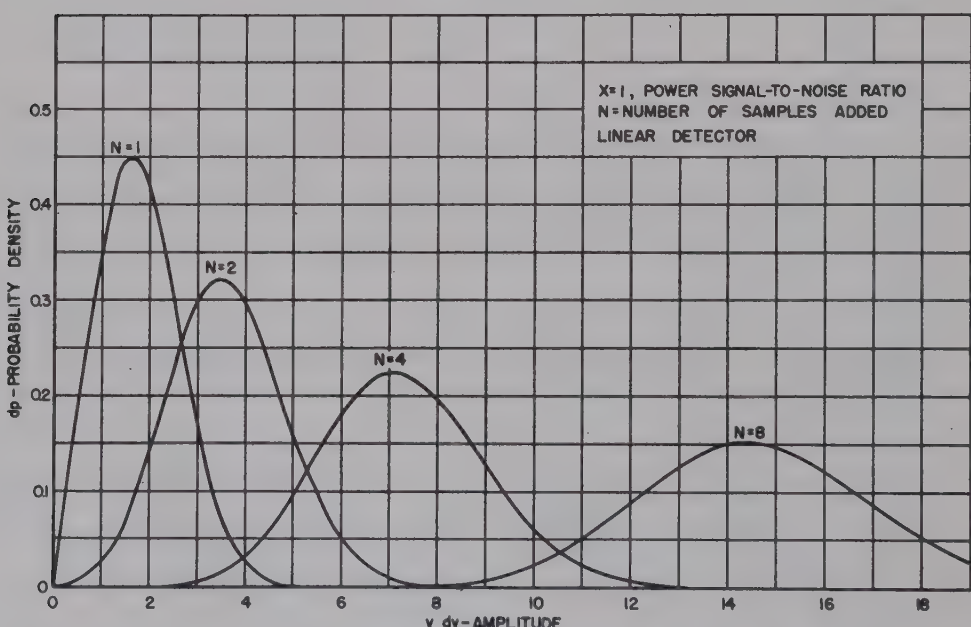
PROBABILITY DENSITY FUNCTIONS FOR ENVELOPE OF SINE WAVE PLUS NOISE

FIG. 39



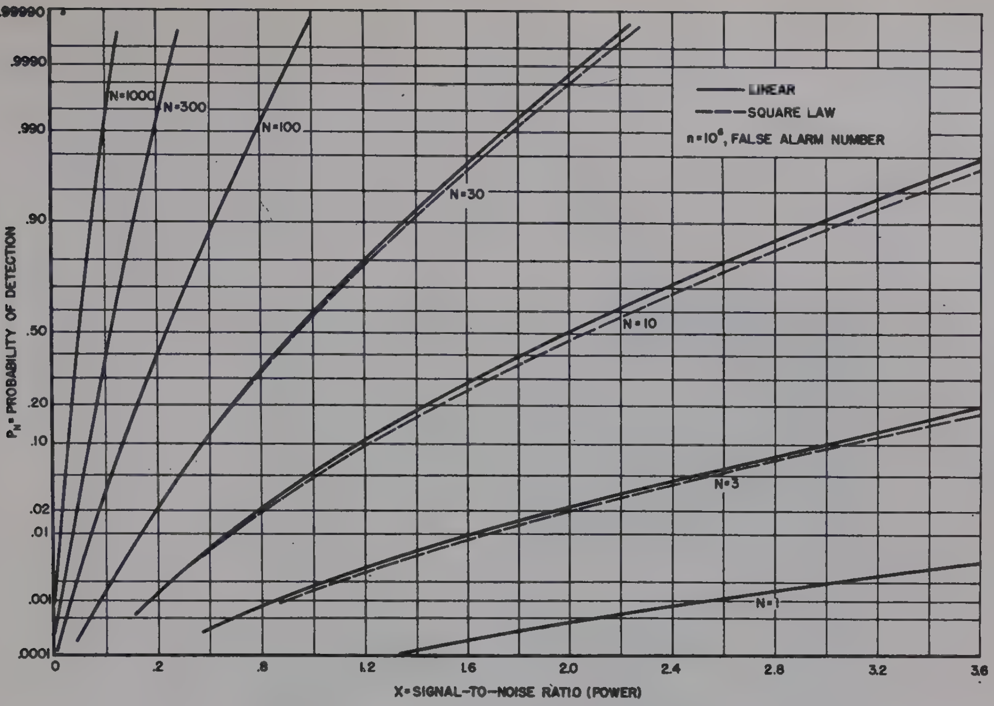
PROBABILITY DENSITY FUNCTIONS FOR ENVELOPE OF SINE WAVE PLUS NOISE

FIG. 40



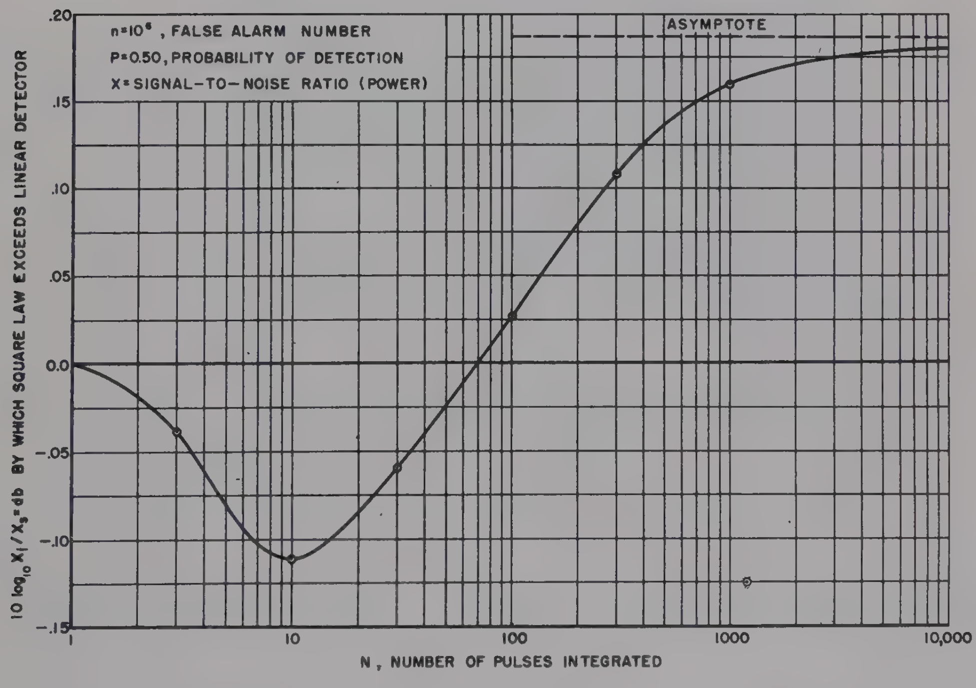
PROBABILITY DENSITY FUNCTIONS FOR ENVELOPE OF SINE WAVE PLUS NOISE

FIG. 41



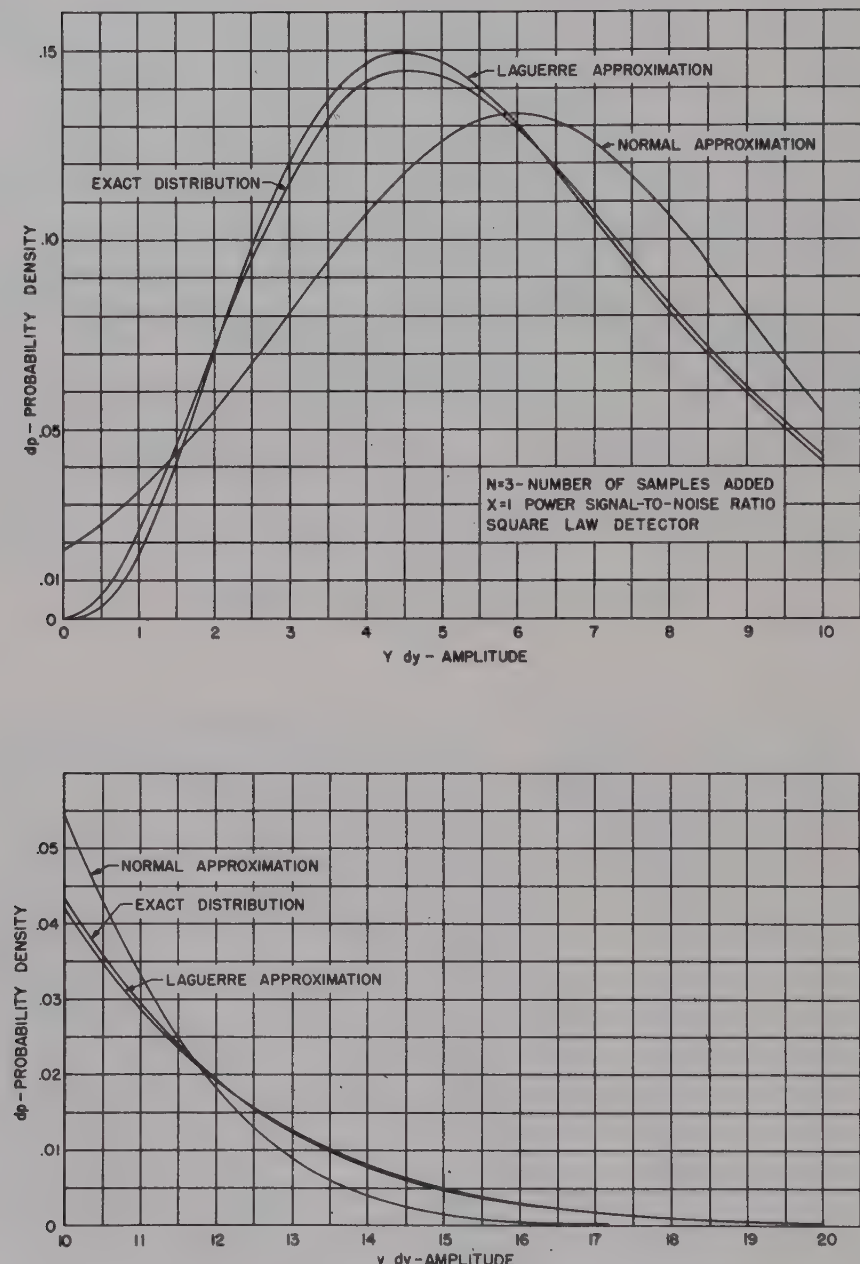
COMPARISON OF LINEAR AND SQUARE LAW DETECTORS

FIG. 42



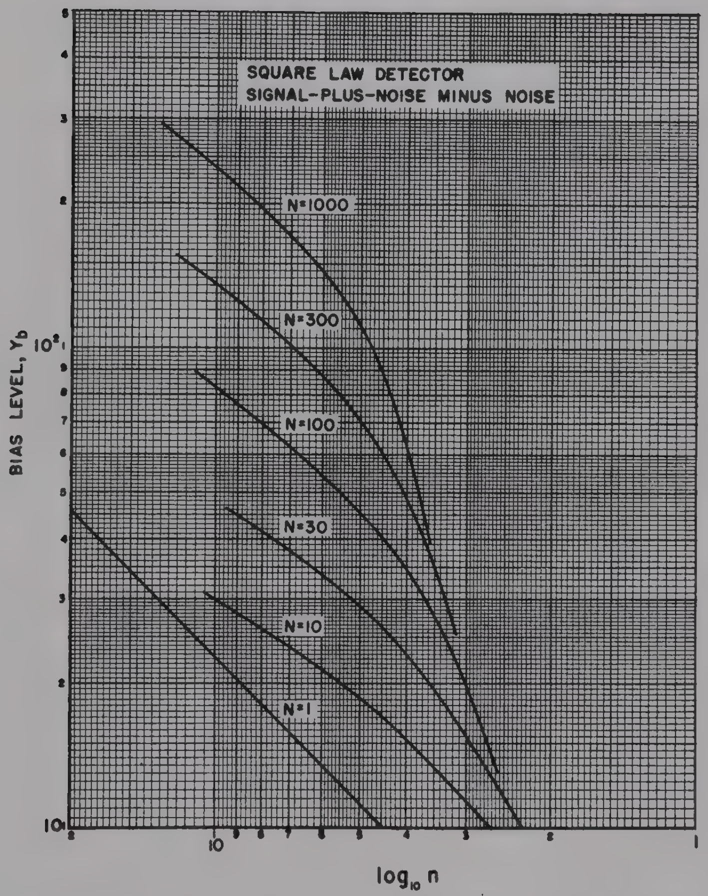
COMPARISON OF LINEAR AND SQUARE LAW DETECTORS

FIG. 43



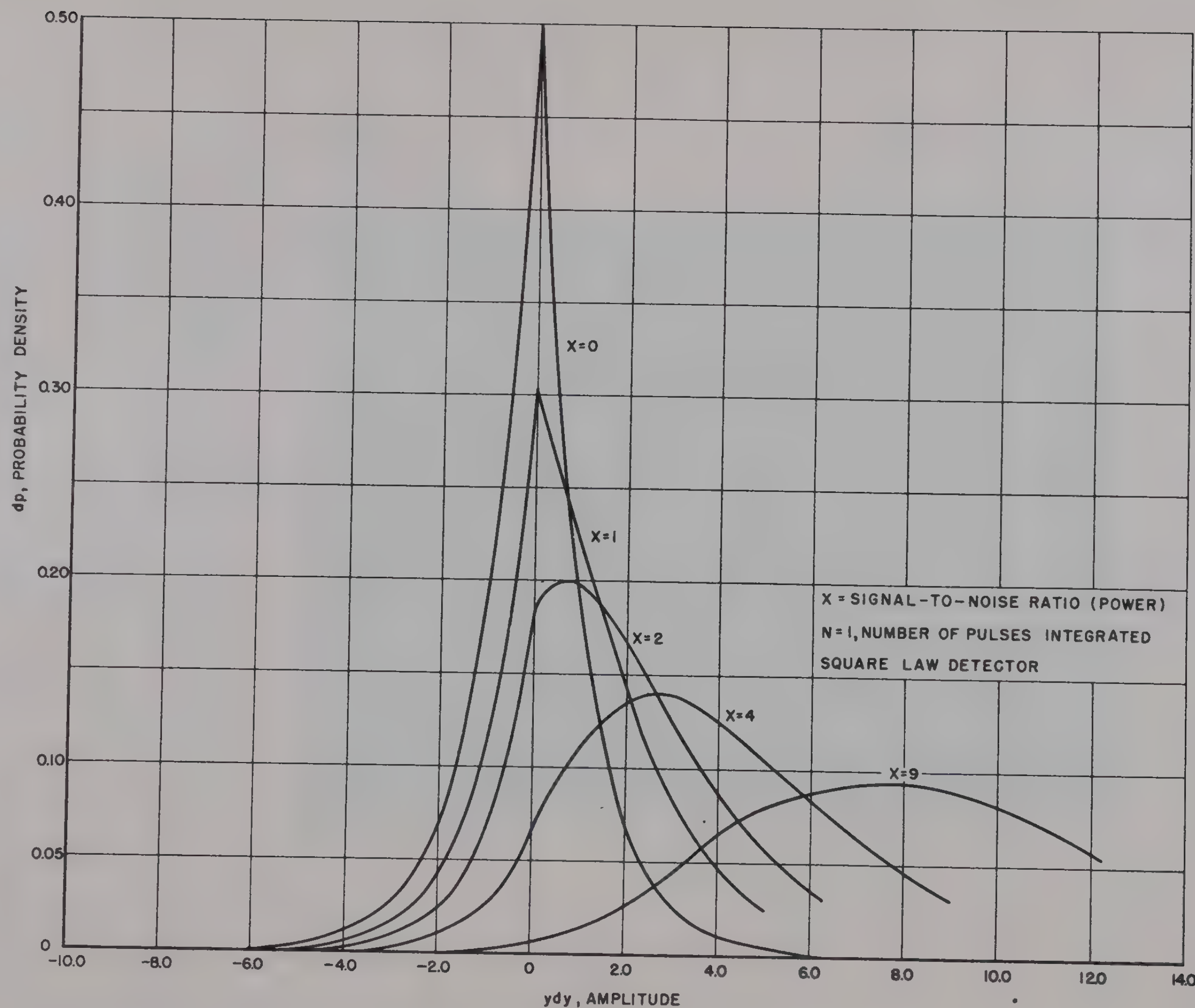
FIRST APPROXIMATIONS TO THE PROBABILITY DENSITY FOR SIGNAL PLUS NOISE

FIG. 44

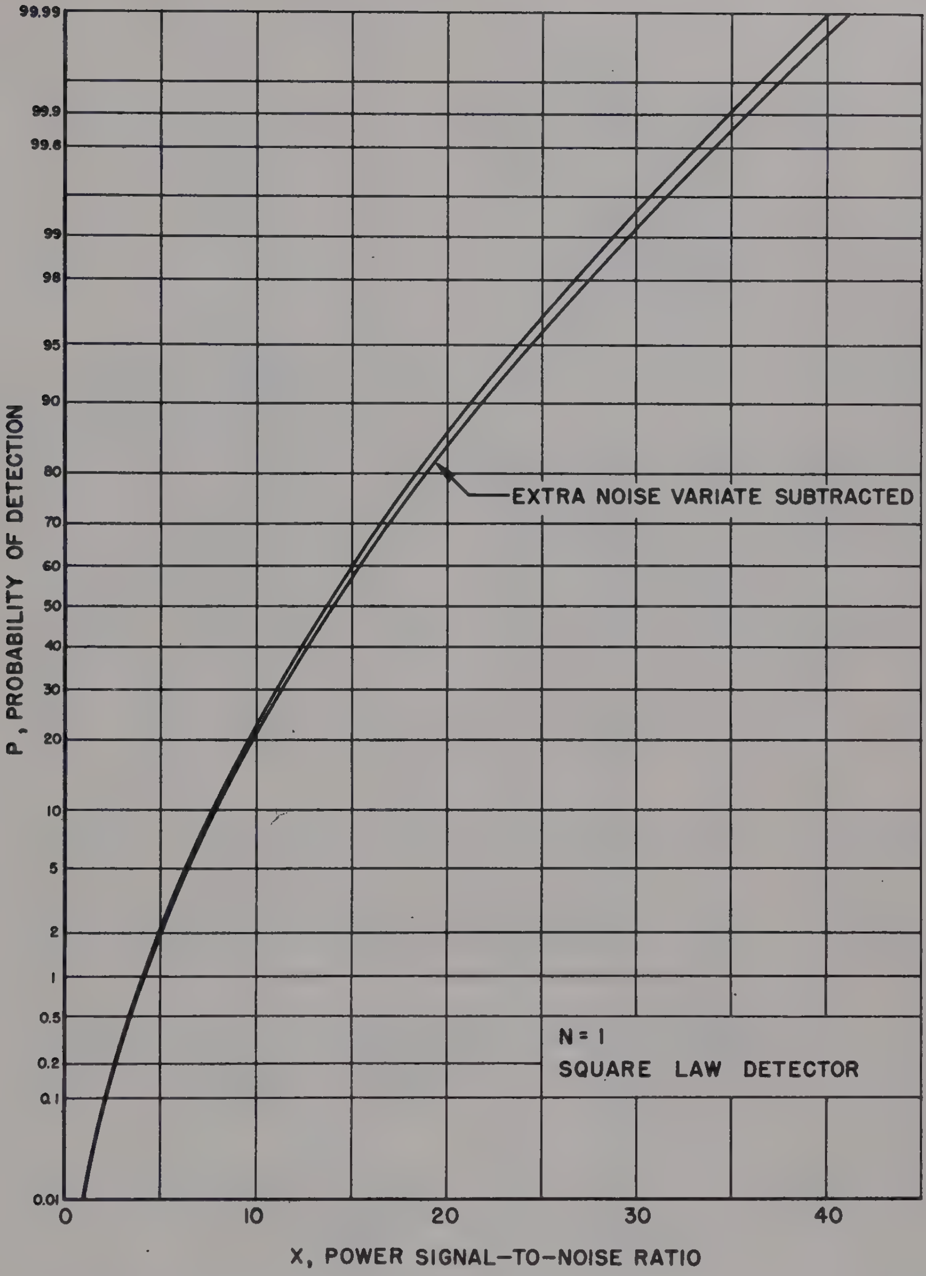


BIAS LEVEL AS A FUNCTION OF NUMBER OF PULSES INTEGRATED AND FALSE ALARM NUMBER

FIG.45

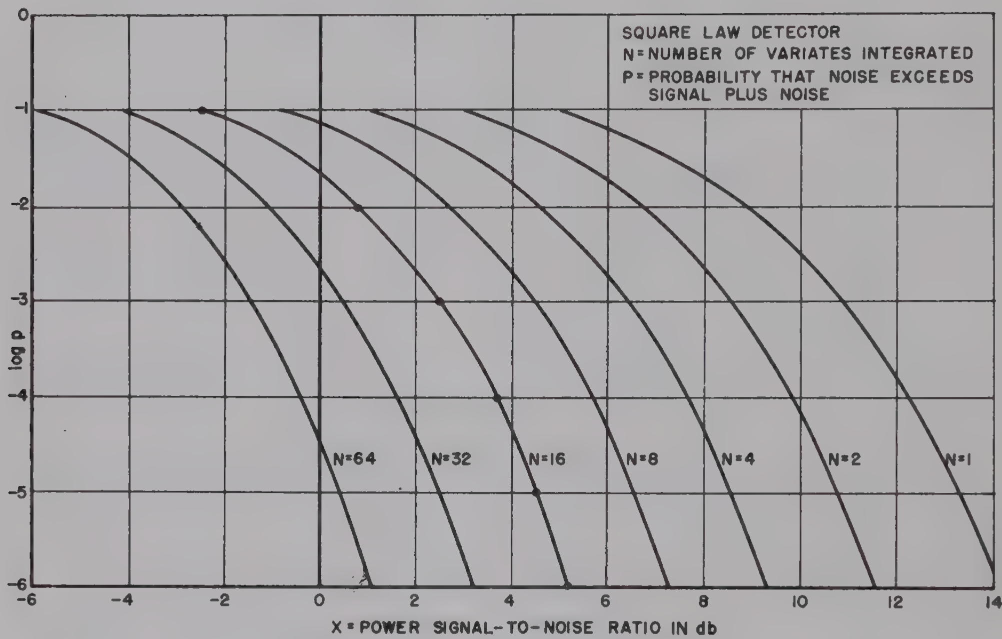


PROBABILITY DENSITY FUNCTIONS FOR A SIGNAL-PLUS-NOISE VARIATE MINUS A NOISE VARIATE
FIG. 46



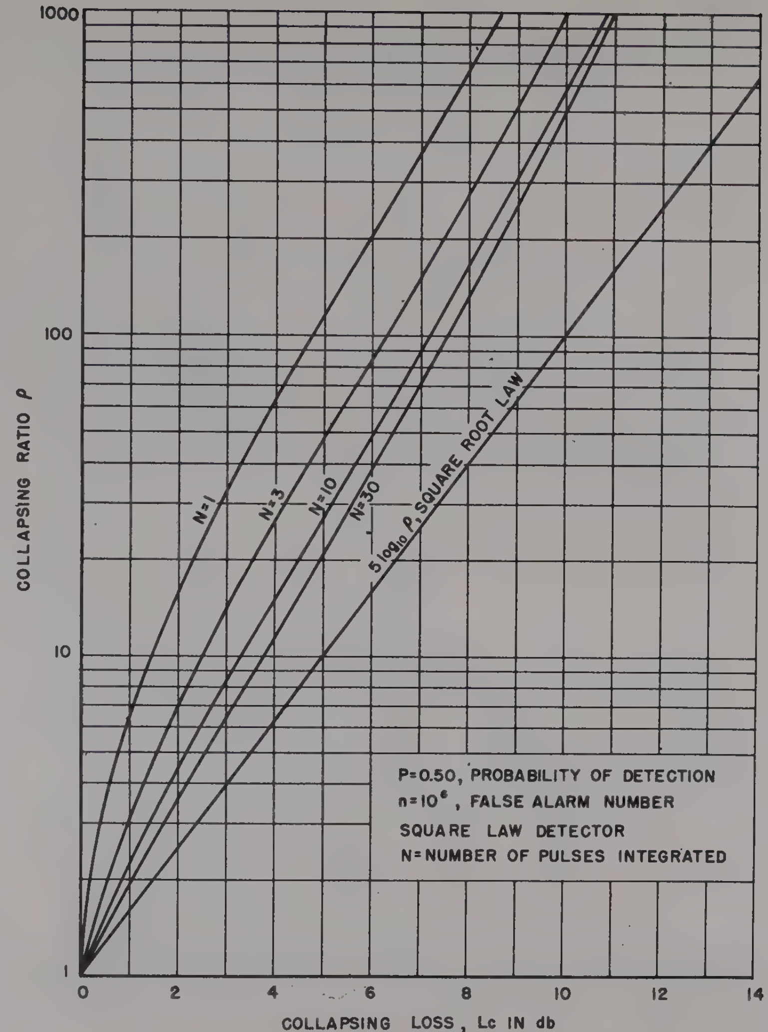
COMPARISON OF PROBABILITY OF DETECTION WHEN A NOISE VARIATE IS SUBTRACTED FROM THE SIGNAL-PLUS-NOISE VARIATE

FIG.47



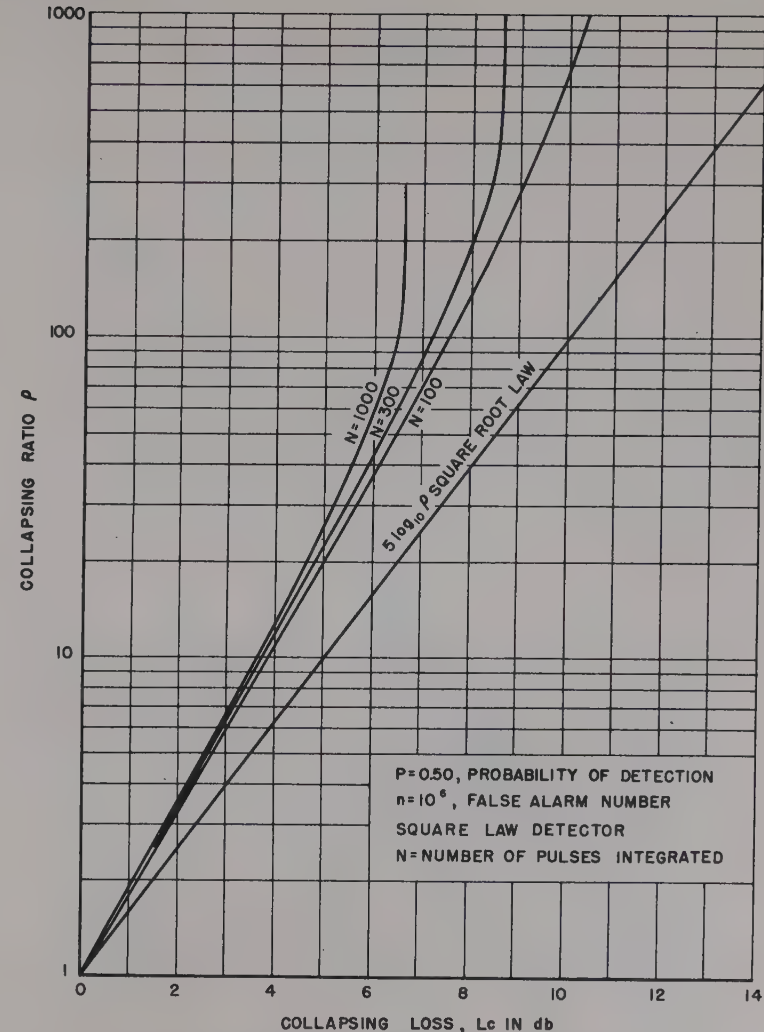
PROBABILITLY THAT NOISE EXCEEDS SIGNAL-PLUS-NOISE

FIG. 48



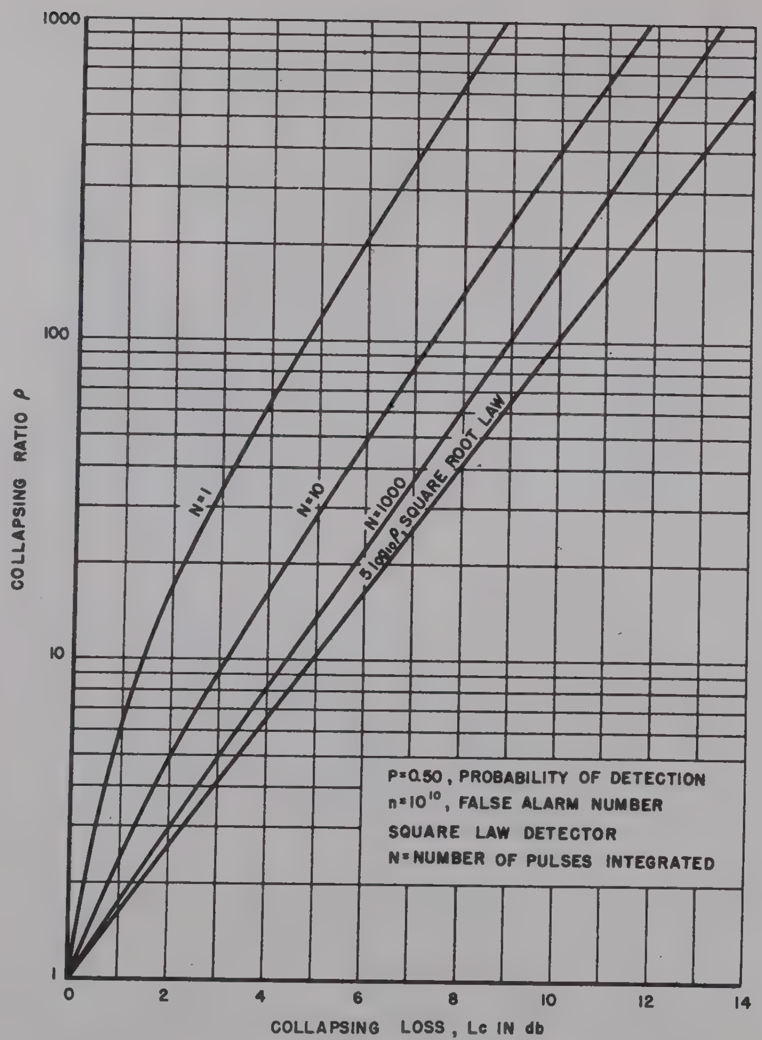
COLLAPSING LOSS FOR CONSTANT n

FIG.49



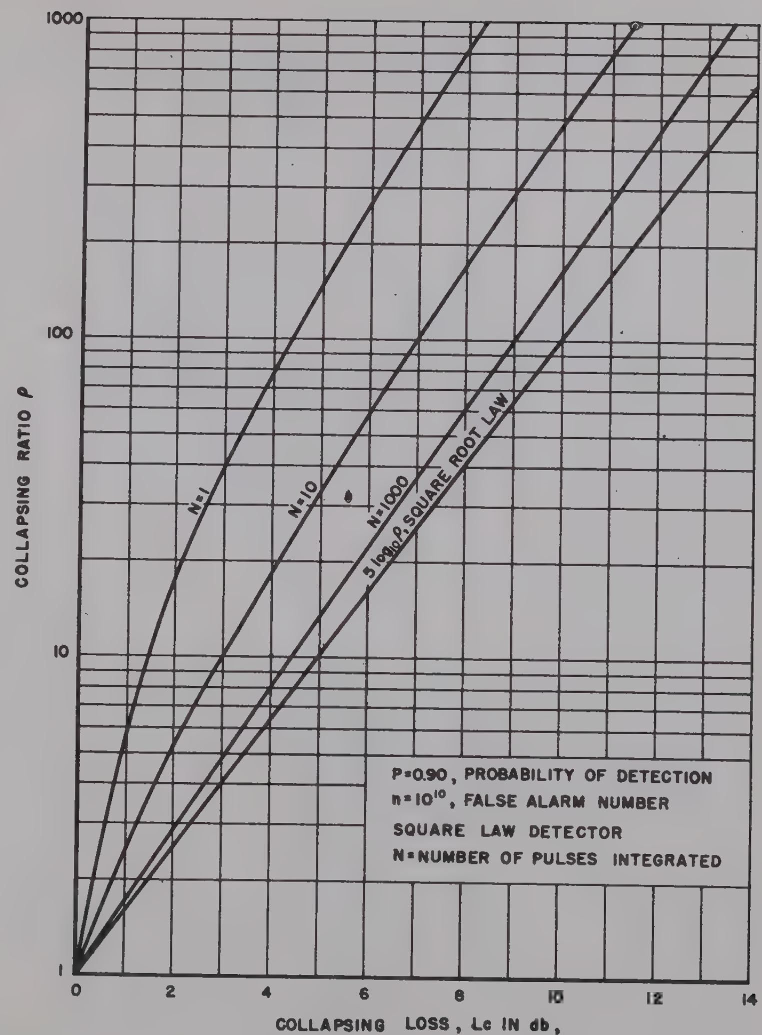
COLLAPSING LOSS FOR CONSTANT n

FIG.50



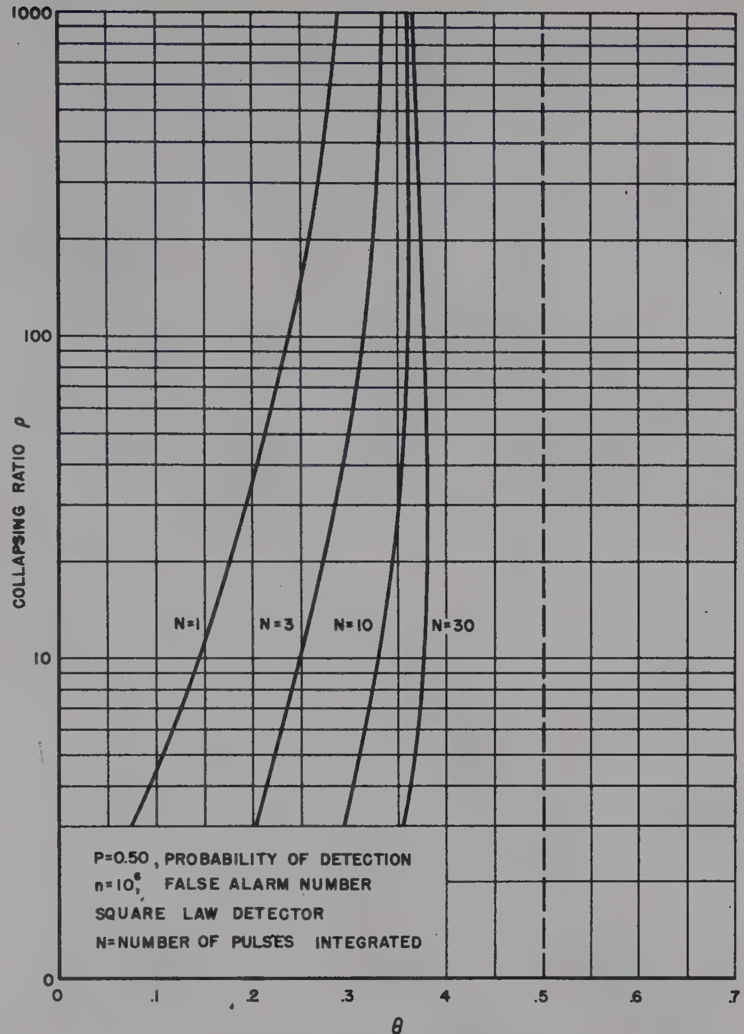
COLLAPSING LOSS FOR CONSTANT n

FIG. 51

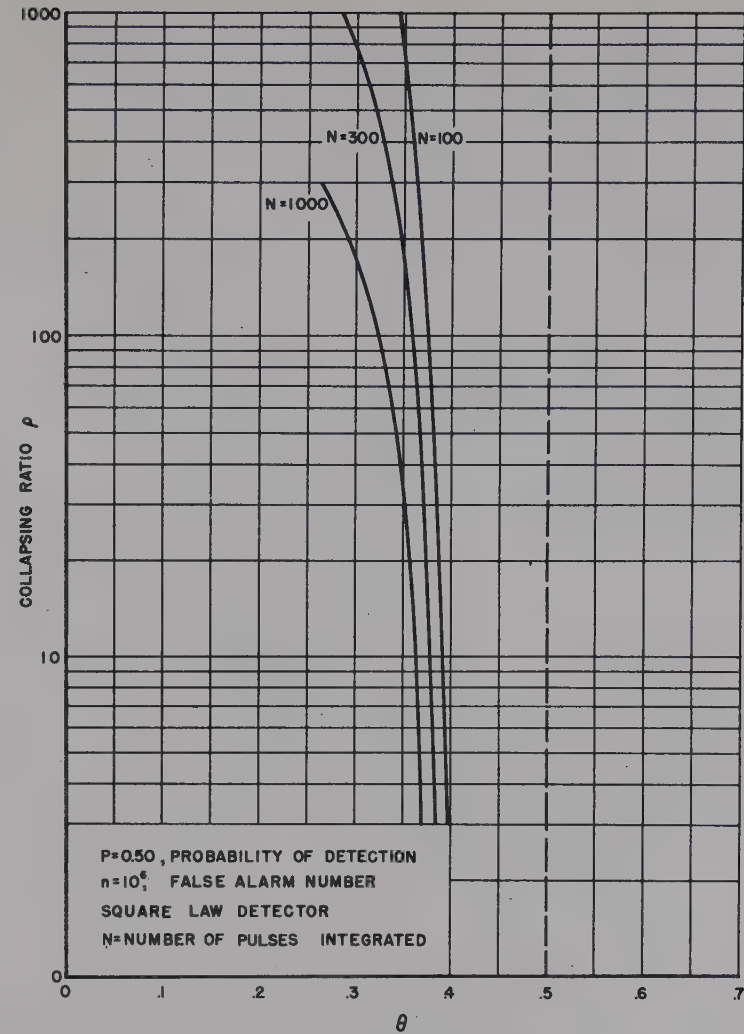


COLLAPSING LOSS FOR CONSTANT n

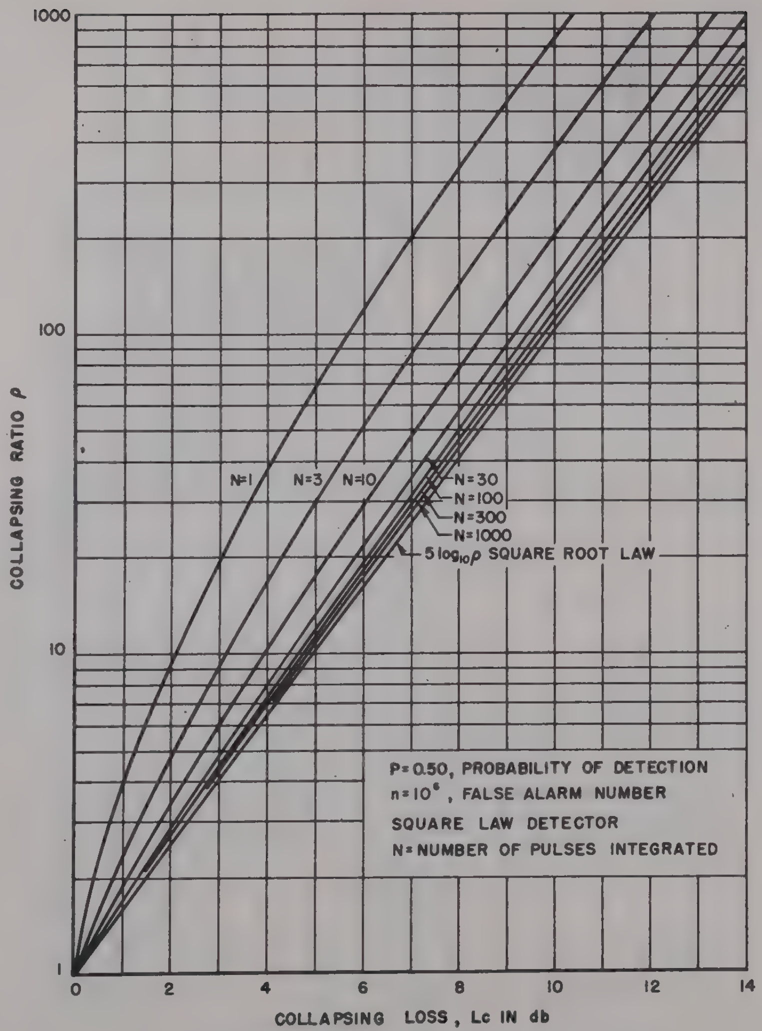
FIG. 52



COLLAPSING LOSS FOR CONSTANT n
FIG.53

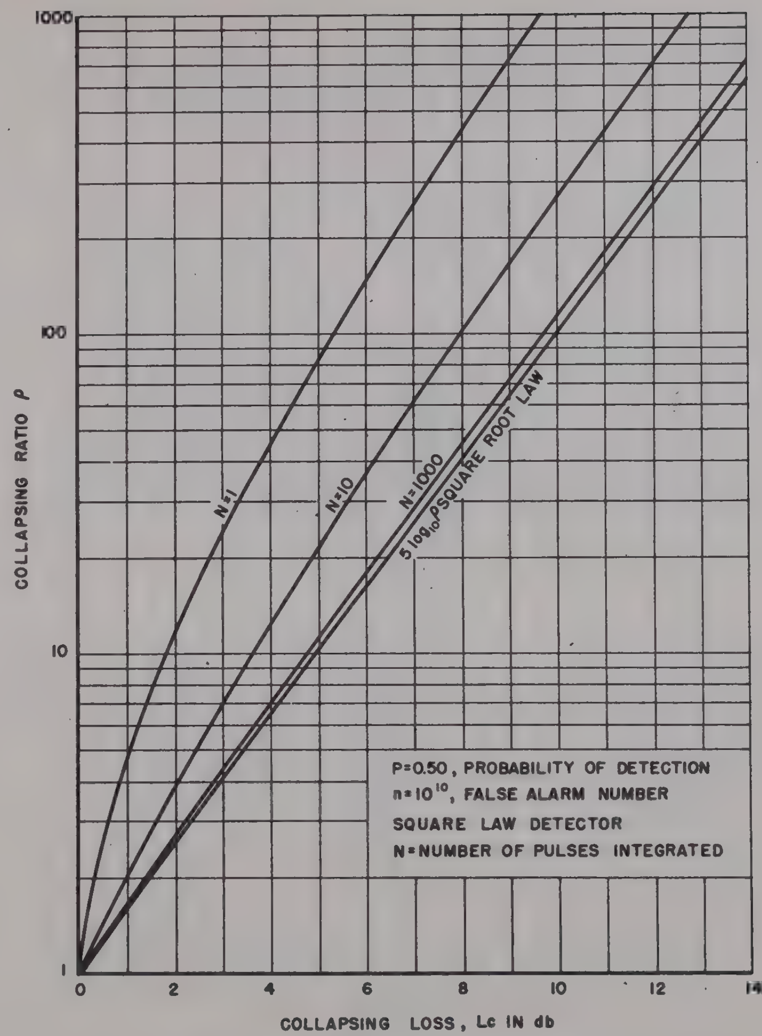


COLLAPSING LOSS FOR CONSTANT n
FIG.54



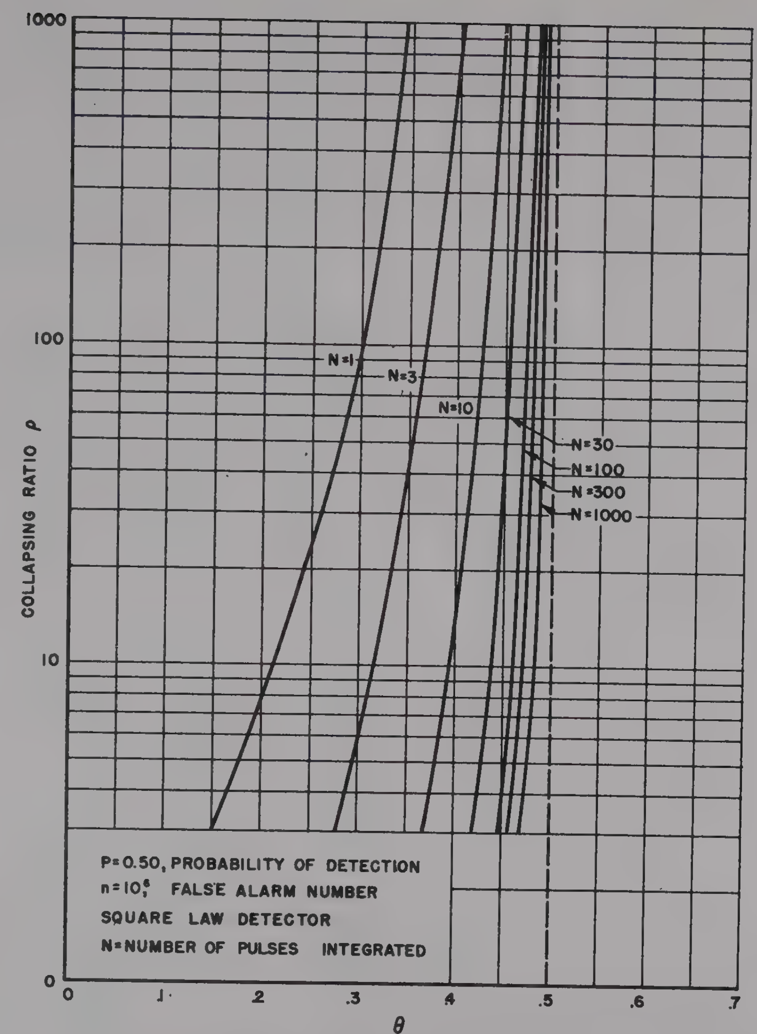
COLLAPSING LOSS FOR CONSTANT n'

FIG. 55

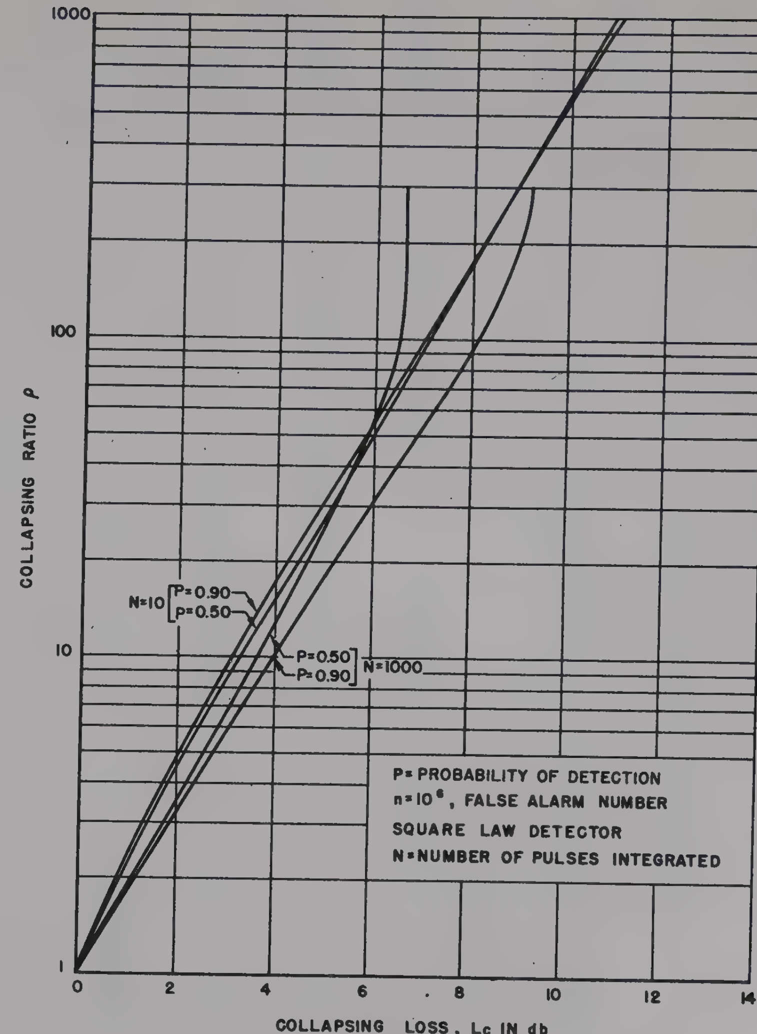


COLLAPSING LOSS FOR CONSTANT n'

FIG. 56



COLLAPSING LOSS FOR CONSTANT n'
FIG. 57



COLLAPSING LOSS FOR CONSTANT n
FIG. 58

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Probability of Detection for Fluctuating Targets

(RM-1217, March 17, 1954)

by

P. Swerling

CONTENTS

SUMMARY

This report considers the probability of detection of a target by a pulsed search radar, when the target has a fluctuating cross section.

Formulas for detection probability are derived, and curves of detection probability vs. range are given, for four different target fluctuation models.

The investigation shows that, for these fluctuation models, the probability of detection for a fluctuating target is less than that for a non-fluctuating target if the range is sufficiently short, and is greater if the range is sufficiently long.

The amount by which the fluctuating and non-fluctuating cases differ depends on the rapidity of fluctuation and on the statistical distribution of the fluctuations. Figure 18, p.307, shows a comparison between the non-fluctuating case and the four fluctuating cases considered.

SYMBOLS

$f(n,N)$	scale factor, Case 3
$g(n,N)$	scale factor, Case 1
I	incomplete gamma function
N	number of hits integrated
n	false alarm number
P_D	probability of detection
R	range
R_o	range for which average input signal-to-noise ratio equals unity
x	input signal-to-noise power ratio for a single pulse
$\bar{x}$	average of x over all target fluctuations
Y_b	normalized threshold

I. INTRODUCTION

The probability of detection of a target by a pulsed search radar has been treated in considerable detail by J. I. Marcum[1,2] for the case in which the amplitude of the returned signal pulses is not fluctuating. The purpose of this paper is to extend some of Marcum's results - mainly the computation of probability of detection* vs. range curves - to several kinds of fluctuating targets. (No claim to originality is made. Several of the equations given below have been derived by Marcum in hitherto unpublished work; some of these equations have also appeared elsewhere in the literature. This report is for the purpose of making the results more readily available than heretofore.) A general familiarity, on the part of the reader, with Marcum's paper will be assumed. Marcum's notation will be used throughout.

Four different models of target fluctuation will be considered. The four models chosen for consideration are felt to be typical of situations which are likely to be met in practice, or, at least, to bracket a wide range of practical cases.

In applying probability of detection computations to actual cases, one should first attempt to analyze the fluctuations of the actual target under consideration, and then choose whichever model (including Marcum's non-fluctuating model) appears to most closely approximate the actual target fluctuations. Or, one could consider the actual target to be intermediate between two of the theoretical models.

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* Use is made of the term 'probability of detection' in order to conform to Marcum's terminology. Actually 'blip-scan ratio' is a more common term to use for the quantity which is computed. (The curves presented here all correspond to the cases which in Marcum's paper are labeled $\gamma = N$.)

One of the uses of the ensuing results is, that by comparing the results for different models, one can make some judgment as to the errors introduced by choosing the wrong fluctuation model.

The four fluctuation models considered are as follows:

CASE 1

The returned signal power per pulse is assumed to be constant for the time on target during a single scan, but to fluctuate independently from scan to scan. (This ignores factors such as beam shape effect.) Expressed in statistical terms, the normalized autocorrelation function of target cross section is approximately one for the time in which the beam is on target during a single scan, and is approximately zero for a time as long as the interval between scans. This type of fluctuation will henceforth be referred to as scan-to-scan fluctuation.

The fluctuations of target cross section are evidenced as fluctuations of signal to noise ratio in the receiver. The probability density for the input signal-to-noise power ratio is assumed to be:

$$w(x,\bar{x}) = \frac{1}{\bar{x}} e^{-x/\bar{x}} \quad (x \geq 0) \; * \qquad (I.1)$$

where x = input signal-to-noise power ratio

$\bar{x}$ = average of x over all target fluctuations

CASE 2

The fluctuations are independent from pulse to pulse. This type of fluctuation will be referred to as pulse-to-pulse fluctuation.

The probability density function is given by (I.1)

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* This formula also represents the probability density for target cross section Σ if x is replaced by Σ and $\bar{x}$ by $\bar{\Sigma}$.

CASE 3

Scan-to-scan fluctuation according to the probability density

$$w(x,\bar{x}) = \frac{4x}{\bar{x}^2} e^{-2x/\bar{x}} \quad (x \geqq 0) \qquad (I.2)$$

CASE 4

Pulse-to-pulse fluctuation according to (I.2).

It would be well at this point to indicate in which actual situations the various models would be likely to apply.

As to the choice of probability density function for the fluctuations: Theoretically, for a target which can be represented as several independently fluctuating reflectors of approximately equal echoing area, the density function should be close to exponential, even if the number of reflectors is as small as four or five. Thus one would expect objects which are large compared with wavelength (and which are not shaped too much like a sphere) to fluctuate approximately according to the exponential density (I.1).

On the other hand, a target which can be represented as one large reflector together with other small reflectors, or as one large reflector subject to fairly small changes in orientation, would be expected to behave more like (I.2).

The non-fluctuating model would apply to spherical or nearly spherical objects (e.g. balloons, meteors) at fairly large wavelengths.

Most available observational data on aircraft targets indicates agreement with the exponential density (I.1). More definite statements as to actual targets for which (I.2) or the non-fluctuating model apply must await further experimental data.

As to the choice between scan-to-scan and pulse-to-pulse fluctuation:

The scan-to-scan model would apply to targets such as jet aircraft or missiles, for radars having fairly high pulse repetition rate and scan rate.

Pulse-to-pulse fluctuation would apply to propeller-driven craft if the propellers contribute a large portion of the echoing area; or to targets for which very small changes in orientation would mean large changes in echoing area, such as long, thin objects at high frequency; or to targets viewed by a radar with sufficiently low repetition rate.

Most actual targets would probably be intermediate between the various cases considered.

A comparison between Cases 1, 2, 3, 4, and the non-fluctuating case is given in Fig. 18 (for typical false alarm time and number of hits integrated).

In all cases, it is assumed that there are on each scan N hits; after passage through a square law second detector, the resulting N pulses are added and required to exceed a threshold Y_b in order for detection of a target to occur.* The second detector output is, for mathematical convenience, assumed to be normalized as follows: detector output equals input envelope squared divided by twice the mean square input noise voltage.

Formulas for probability of detection P_D as a function of $\bar{x}$ are given for each case in Section II. Curves (corresponding to Marcum's) for P_D vs range are given in Figs. 1-18. (Marcum's full range of parameters is not duplicated, but the means for doing so are given by the formulas in Section II.) Since the formulas in Cases 1 and 3 lend themselves to very convenient approximations, these cases are further discussed in Section II. The derivations of the formulas in Section II are given in Section III.

* The actual beam shape is in effect being approximated by a beam having uniform gain over a finite sector, and zero gain outside this sector. In principle it is possible to take account exactly of beam shape in computing probability of detection. This is not done here, however, since it is thought that the effect of the aforementioned approximation is small provided the effective number N of hits per scan and effective $\bar{x}$ are properly chosen.

II. FORMULAS FOR P_D

CASE 1

The exact formula is

$$N = 1: \quad P_D = \exp\left[\frac{-Y_b}{1+\bar{x}}\right]$$

$$N > 1: \quad P_D = 1 - I\left[\frac{Y_b}{\sqrt{N-1}},\ N-2\right]$$

$$+ \left(1 + \frac{1}{N\bar{x}}\right)^{N-1} I\left[\frac{Y_b}{\left(1+\frac{1}{N\bar{x}}\right)\sqrt{N-1}},\ N-2\right] \exp\left[\frac{-Y_b}{1+N\bar{x}}\right] \qquad \text{(II.1)}$$

where I is the incomplete gamma function.(3)

In most cases, this can be closely approximated by

$$P_D \approx \left(1 + \frac{1}{N\bar{x}}\right)^{N-1} \exp\left[\frac{-Y_b}{1+N\bar{x}}\right] \qquad \text{(II.2)}$$

For $N = 1$, (II.2) is exact.

For $N > 1$, in most cases of interest, $N\bar{x}$ is several times greater than one. If this is true, and if the false alarm probability is sufficiently small, then, for $N > 1$, the gamma function factors are very nearly unity. It turns out that in most cases of interest, the values of $N\bar{x}$ and false alarm probability are such that (II.2) can be regarded as practically exact. (One must keep in mind, of course, that the assumptions $N\bar{x} > 1$ and negligible false alarm probability mean that one cannot use (II.2) for indefinitely small values of P_D. In most cases of interest, the applicability of (II.2) extends at least down to P_D = one per cent.)

Now, for $N\bar{x} > 1$ the expansion of $\ln P_D$ as given by (II-2) is

$$\ln P_D = -\frac{1}{N\bar{x}}\left(Y_b - N + 1\right) + \frac{1}{(N\bar{x})^2}\left(Y_b - \frac{N-1}{2}\right) - \frac{1}{(N\bar{x})^3}\left(Y_b - \frac{N-1}{3}\right) + \ldots \qquad \text{(II.3)}$$

This is a convenient way in which to compute P_D. Curves of P_D vs $\frac{R}{R_o}$ based on (II.2) are given in Figs. 1 - 3.

A good approximation to P_D can be obtained by using only the first term of this expansion, giving

$$P_D \approx \exp\left[\frac{-h(n,N)}{\bar{x}}\right] \tag{II.4}$$

where

$$h(n,N) = \frac{Y_b - N + 1}{N} \tag{II.5}$$

Here n is the false alarm number; Y_b is a function of both N and n.*

Since $\frac{1}{\bar{x}} = \left(\frac{R}{R_o}\right)^4$, this leads to the result

$$P_D = e^{-u^4}$$

$$\frac{R}{R_o} = \frac{u}{g(n,N)} \tag{II.6}$$

and

$$g(n,N) = \left(\frac{Y_b - N + 1}{N}\right)^{1/4} \tag{II.7}$$

In other words, the P_D vs $\frac{R}{R_o}$ curves are all, to a good approximation, simply exp $(-u^4)$, with $\frac{R}{R_o}$ = u·scale factor $1/g(n,N)$. Curves of exp $(-u^4)$ and $g(n,N)$ are given in Figs. 4 and 5.

To see approximately the error introduced in going from (II.2) to (II.6): for each P_D let R_1 be the range computed by using just the first term of the expansion (II.3); let R_2 be the range computed by using the first two terms. Then**

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* Curves of Y_b vs N and n are to be found in Ref. 2. In most cases, n is approximately equal to the false alarm time divided by the pulse width.
** See Appendix A for derivation.

$$\frac{R_2}{R_1} \approx 1 - \mathcal{E}(n,N) \ln P_D \tag{II.8}$$

where

$$\mathcal{E}(n,N) = \frac{Y_b - \frac{N-1}{2}}{4(Y_b-N+1)^2} \tag{II.9}$$

Curves of $\mathcal{E}(n,N)$ are given in Fig. 6.

This is valid for $-16\ \mathcal{E}(n,N) \ln P_D$ less than about .5. Referring to the curves of $\mathcal{E}(n,N)$ in Fig. 6, this means for P_D greater than .01 - .10, depending on the values of n and N. In almost all such cases, the first two terms of (II.3) are the only significant ones, so that (II.8) can be regarded as a correction between (II.6) and (II.2).

The curve $P_D = \exp(-u^4)$ is not necessarily the best curve to use in (II.6) to give the best numerical results in approximating (II.2). It is possible to give this curve an average correction so as to obtain better results over the interesting range of the parameters n and N.

Such a corrected curve, to be used with the scale factor g(n,N) to approximate (II.2), is given in Fig. 4. The use of this curve with g(n,N) gives agreement with (II.2) to within about five per cent in $\frac{R}{R_o}$ for $P_D \geqq .01$, $10^6 \leqq n \leqq 10^{12}$, and $1 \leqq N \leqq 1000$. The agreement gets better as P_D gets larger, and is practically exact for $P_D \geqq .50$.

CASE 2

The exact formula for P_D is

$$P_D = 1 - I\left[\frac{Y_b}{(1+\bar{x})\sqrt{N}}, N-1\right] \tag{II.10}$$

Reference 3 contains tables of the incomplete gamma function enabling the computation of (II.10) for $N - 1 \leqq 50$. Beyond this point an Edgeworth series can be used to compute P_D:

$$P_D \approx \frac{1}{2}\left[1-\phi^{-1}(T)\right] - C_3\phi^{(2)}(T) - C_4\phi^{(3)}(T) - C_6\phi^{(5)}(T) \qquad \text{(II.11)}$$

where

$$\phi(t) = \frac{1}{\sqrt{2\pi}} \exp\left[\frac{-t^2}{2}\right]$$

$$\phi^{-1}(T) = \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} \exp\left(\frac{-t^2}{2}\right) dt$$

$$\phi^{(2)}, \phi^{(3)}, \phi^{(5)} \text{ are derivatives of } \phi \qquad \text{(II.12)}$$

and

$$T = \frac{Y_b - N(1+\bar{x})}{\sqrt{N}\ (1+\bar{x})}$$

$$C_3 = \frac{1}{3\sqrt{N}}\ ;\quad C_4 = \frac{1}{4N}\ ;\quad C_6 = \frac{1}{18N} \qquad \text{(II.13)}$$

Curves of P_D vs $\frac{R}{R_o}$ for this case are given in Figs. 7 - 9.

CASE 3

The exact formula in this case is so cumbersome as not to be worth writing out here. (The exact density function is given in Eq. (III.18).)

The formula obtained by assuming the negligibility of false alarm probability, and $\frac{N\bar{x}}{2} > 1$, is

$$P_D \approx \left(1 + \frac{2}{N\bar{x}}\right)^{N-2} \left[1 + \frac{Y_b}{1 + \frac{N\bar{x}}{2}} - \frac{2}{N\bar{x}}(N-2)\right] \exp\left[\frac{-Y_b}{1 + \frac{N\bar{x}}{2}}\right] \qquad \text{(II.14)}$$

This turns out to be the exact formula for N = 1 and N = 2. Curves of P_D vs $\frac{R}{R_o}$ based on (II.14) are given in Figs. 10 - 12.

A close approximation to (II.14) is given by

$$P_D \approx \left[1 + \frac{2}{\bar{x}}\left(\frac{Y_b - N + 2}{N}\right)\right] \exp\left[-\frac{2}{\bar{x}}\left(\frac{Y_b - N + 2}{N}\right)\right] \qquad \text{(II.15)}$$

and since $\frac{1}{\bar{x}} = \left(\frac{R}{R_o}\right)^4$, this gives

$$P_D \approx \left(1 + 2u^4\right) \exp\left[-2u^4\right] \qquad \text{(II.16)}$$

$$\frac{R}{R_o} = \frac{u}{f(n,N)}$$

where

$$f(n,N) = \left(\frac{Y_b - N + 2}{N}\right)^{1/4} \qquad \text{(II.17)}$$

The scale factor $f(n,N)$ is almost indentical to the factor $g(n,N)$ of Case 1, the maximum difference being about 2 per cent for small N, and negligible for moderately large N. Curves of $\left(1 + 2u^4\right) \exp\left[-2u^4\right]$ and $f(n,N)$ are given in Figs. 13 and 14.

Actually, over the interesting range of parameters, the curve $P_D = \left(1 + 2u^4\right) \exp\left[-2u^4\right]$ is not the best curve to use in (II.16) to give the best numerical results in approximating (II.14). It is possible to give this curve an average correction so as to obtain better results over the interesting range of n and N.

Such a corrected curve, to be used with the scale factor $f(n,N)$ to approximate (II.14), is given in Fig. 13. The use of this curve with $f(n,N)$ gives agreement with (II.14) to within about seven per cent in $\frac{R}{R_o}$ for $P_D \geq .01$, $10^6 \leq n \leq 10^{12}$, and $1 \leq N \leq 1000$. The agreement gets better as P_D gets larger, and is practically exact for $P_D \geq .50$.

CASE 4

The exact density function f(v) is a polynomial multiplied by $\exp\left[{}^{-v}\!\Big/\!\left(1 + \frac{\bar{x}}{2}\right)\right]$. Except for small N, the polynomial is too long to be of much use for computation. Hence an Edgeworth series seems to be the best way to compute P_D except for small N.

For N = 1, the exact formula is

$$P_D = \left(1 + \frac{2}{\bar{x}}\right)^{-1} \left[1 + \frac{2}{\bar{x}} + \frac{Y_b}{1 + \frac{\bar{x}}{2}}\right] \exp\left[\frac{-Y_b}{1 + \frac{\bar{x}}{2}}\right] \qquad \text{(II.18)}$$

For N moderately large, the Edgeworth series is given by (II.11) with, letting $\beta = 1 + \frac{\bar{x}}{2}$,

$$T = \frac{Y_b - N(1 + \bar{x})}{\sqrt{N(2\beta^2 - 1)}}$$

$$C_3 = \frac{1}{3\sqrt{N}} \frac{(2\beta^3 - 1)}{(2\beta^2 - 1)^{3/2}}$$

$$C_4 = \frac{1}{4N} \frac{(2\beta^4 - 1)}{(2\beta^2 - 1)^2}$$

$$C_6 = \frac{1}{18N} \frac{(2\beta^3 - 1)^2}{(2\beta^2 - 1)^3} \qquad \text{(II.19)}$$

Curves of P_D vs $\frac{R}{R_o}$ for this case are given in Figs. 15 - 17.

III. DERIVATION OF FORMULAS FOR P_D

The characteristic function for the sum, v, of N normalized pulses of signal plus noise, with constant signal-to-noise power ratio x, for a square law detector, is*

$$C(p) = \frac{\exp\left[-Nx\left(\frac{p}{p+1}\right)\right]}{(p+1)^N} \qquad \text{(III.1)}$$

If x is now a random variable with density function $w(x,\bar{x})$, and x is constant for each group of N pulses but fluctuates independently from group to group, then the characteristic function for the sum of N pulses is, since $w(x,\bar{x}) = 0$ for $x < 0$,

$$\bar{C}(p) = \int_0^\infty w(x,\bar{x})\ C(p)\ dx \qquad \text{(III.2)}$$

CASE 1

Here $w(x,\bar{x}) = \frac{1}{\bar{x}}\ e^{-x/\bar{x}} \quad (x \geq 0)$

therefore

$$\bar{C}(p) = \frac{1}{(1+p)^{N-1}\left[1+p(1+N\bar{x})\right]} \qquad \text{(III.3)}$$

For N = 1, the density function is[4]

$$f(v) = \frac{1}{1+\bar{x}}\quad \exp\left[\frac{-v}{1+\bar{x}}\right] \quad (v \geq 0) \qquad \text{(III.4)}$$

For N > 1, the density function can be found from pair 581.7 of Ref. 4:

$$f(v) = \left(1 + \frac{1}{N\bar{x}}\right)^{N-2} \frac{1}{N\bar{x}}\ I\left[\frac{v}{\left(1+\frac{1}{N\bar{x}}\right)\sqrt{N-1}},\ N-2\right] \exp\left[\frac{-v}{1+N\bar{x}}\right] \qquad \text{(III.5)}$$

therefore, for N = 1,

---

* In the notation used here, if f(v) is the density function and $\zeta(s) = \int f(v)e^{-isv}\,dv$ then $p = is$ and $C(p) = \zeta(s)$.

$$P_D = \exp\left[\frac{-Y_b}{1+\bar{x}}\right] \quad (N = 1) \tag{III.6}$$

For $N > 1$, a simple way to get P_D is:

$$\bar{C}(p) - \frac{1}{(1+p)^{N-1}} = -p\,(1 + N\bar{x})\,\bar{C}(p)$$

so

$$f(v) - \frac{v^{N-2}e^{-v}}{(N-2)!} = -\,(1 + N\bar{x})\,f'(v)$$

and therefore

$$\int_0^{Y_b} f(v)\,dv = I\left[\frac{Y_b}{\sqrt{N-1}},\ N-2\right] - (1 + N\bar{x})\,f(Y_b)$$

Since $P_D = 1 - \int_0^{Y_b} f(v)\,dv$, we have for $N > 1$:

$$P_D = 1 - I\left[\frac{Y_b}{\sqrt{N-1}},\ N-2\right] + \left(1 + \frac{1}{N\bar{x}}\right)^{N-1} I\left[\frac{Y_b}{\left(1 + \frac{1}{N\bar{x}}\right)\sqrt{N-1}},\ N-2\right] \exp\left[\frac{-Y_b}{1 + N\bar{x}}\right] \tag{III.7}$$

<u>CASE 2</u>

The characteristic function for one pulse from a target fluctuating according to (I.1) is (from III.3 for $N = 1$):

$$\bar{C}_1(p) = \frac{1}{1 + p(1 + \bar{x})} \tag{III.8}$$

Therefore the characteristic function for the sum of N pulses, fluctuating independently from pulse to pulse, is

$$\left[\bar{C}_1(p)\right]^N = \frac{1}{\left[1 + p(1 + \bar{x})\right]^N} \tag{III.9}$$

The density function is

$$f(v) = \frac{1}{(1+\bar{x})^N (N-1)!} v^{N-1} \exp\left[\frac{-v}{1+\bar{x}}\right] \tag{III.10}$$

and therefore

$$P_D = 1 - I\left[\frac{Y_b}{(1+\bar{x})\sqrt{N}}, N-1\right] \tag{III.11}$$

Reference 3 contains tables of this function up to $N - 1 = 50$. Beyond this, one can use an Edgeworth series[2] to compute P_D. This can be done if one knows the cumulants K_n for the density function $f(v)$ (as well as the mean).[2]

In this case

$$K_n = N(-1)^{n+1}\left\{\frac{d^n}{dp^n}\left[\ln(1+p(1+\bar{x}))\right]\right\}_{p=0}$$

or

$$K_n = N(n-1)!\ (1+\bar{x})^n \tag{III.12}$$

In addition, it is easily found that the mean of the sum of N pulses is $N(1 + \bar{x})$; the variance is $N(1+\bar{x})^2$. Therefore the coefficients of the second, third, and fourth terms of the Edgeworth series are

$$C_3 = \frac{1}{3\sqrt{N}};\ C_4 = \frac{1}{4N};\ C_6 = \frac{1}{18N} \tag{III.13}$$

<u>CASE 3</u>

Here $w(x,\bar{x}) = \frac{4x}{\bar{x}^2} e^{-2x/\bar{x}}$ and therefore

$$\bar{C}(p) = \frac{1}{(1+p)^{N-2}\left[1 + p\left(1 + \frac{N\bar{x}}{2}\right)\right]^2} \tag{III.14}$$

For N = 1 and N = 2, the density function is obtained from Ref. 4 as:

$$N = 1: \quad f(v) = \frac{1}{\left(1 + \frac{\bar{x}}{2}\right)^2} \left[1 + \frac{v}{1 + \frac{2}{\bar{x}}}\right] \exp\left[\frac{-v}{1 + \frac{\bar{x}}{2}}\right]$$

$$N = 2: \quad f(v) = \frac{1}{(1 + \bar{x})^2} \; v \exp\left[\frac{-v}{1 + \bar{x}}\right] \tag{III.15}$$

From Ref. 4, pair 581.1, after appropriate transformation, one finds that for N > 2,

$$f(v) = \frac{1}{(N-1)! \left(1 + \frac{N\bar{x}}{2}\right)^2} \; v^{N-1} \, e^{-v} \quad {}_1F_1\left[2, N, \; \frac{v}{1 + \frac{2}{N\bar{x}}}\right] \tag{III.16}$$

where ${}_1F_1$ is the confluent hypergeometric function.*

One may now use two identities concerning this hypergeometric function.**

$$\text{a)} \quad {}_1F_1(2,N,z) = (z + 2 - N) \, {}_1F_1 \, (1,N,z) + N - 1$$

$$\text{b)} \quad {}_1F_1(1,N,z) = e^z \, z^{-N+1} \, (N - 1)! \; I\left[\frac{z}{\sqrt{N-1}}, \; N - 2\right] \tag{III.17}$$

to get

$$f(v) = \frac{\left(1 + \frac{2}{N\bar{x}}\right)^{N-2}}{\left(1 + \frac{N\bar{x}}{2}\right)^2} \; v \, I\left[\frac{v}{\left(1 + \frac{2}{N\bar{x}}\right)\sqrt{N-1}}, \; N-2\right] \exp\left[\frac{-v}{1 + \frac{N\bar{x}}{2}}\right]$$

$$- \frac{(N - 2)\left(1 + \frac{2}{N\bar{x}}\right)^{N-1}}{\left(1 + \frac{N\bar{x}}{2}\right)^2} \; I\left[\frac{v}{\left(1 + \frac{2}{N\bar{x}}\right)\sqrt{N-1}}, \; N-2\right] \exp\left[\frac{-v}{1 + \frac{N\bar{x}}{2}}\right]$$

$$+ \frac{1}{\left(1 + \frac{N\bar{x}}{2}\right)^2 (N-2)\,!} \; v^{N-1} \quad e^{-v} \tag{III.18}$$

---

* See Refs. 2 and 4.

** Reference 2, mathematical appendix, pp 20-21. (In this reference there is a misprint in the printing of identity (b)).

If one assumes the false alarm probability to be negligible, and $\frac{N\bar{x}}{2}$ to be >1, then for $v > Y_b$, the I factors are practically equal to one; also, the integral from Y_b to ∞ of the third term in (III.18) can be neglected, for $N > 2$. Therefore, integrating (III.18) from Y_b to ∞, one obtains for $N > 2$:

$$P_D \approx \left(1 + \frac{2}{N\bar{x}}\right)^{N-2} \left[1 + \frac{Y_b}{1 + \frac{N\bar{x}}{2}} - \frac{2(N-2)}{N\bar{x}}\right] \exp\left[\frac{-Y_b}{1 + \frac{N\bar{x}}{2}}\right] \qquad \text{(III.19)}$$

This is the exact formula for N = 1 and N = 2, as can be found directly from (III.15).

CASE 4

For one pulse (setting N = 1 in (III.14))

$$\bar{C}_1(p) = \frac{1 + p}{\left[1 + p\left(1 + \frac{\bar{x}}{2}\right)\right]^2} \qquad \text{(III.20)}$$

Therefore for the sum of N pulses, fluctuating independently from pulse to pulse, the characteristic function is

$$\left[\bar{C}_1(p)\right]^N = \frac{(1 + p)^N}{\left[1 + p\left(1 + \frac{\bar{x}}{2}\right)\right]^{2N}} \qquad \text{(III.21)}$$

The exact density function is obtainable and turns out to be a polynomial multiplied by $\exp\left[{}^{-v}\Big/\left(1 + \frac{\bar{x}}{2}\right)\right]$. Except for small N, this polynomial is too long to be useful for computational purposes. For fairly large N, one can use the Edgeworth series. For N = 1, the exact formula is, of course, given by (III.15) with N = 1.

The cumulants, obtained in the usual way, are

$$K_n = N(n-1)!\left[2\beta^n - 1\right] \text{ where } \beta = 1 + \frac{\bar{x}}{2} \qquad \text{(III.22)}$$

Also, mean = $N(1 + \bar{x})$

variance = $N(2\beta^2-1)$

So, the coefficients for the second, third, and fourth terms of the Edgeworth series are

$$C_3 = \frac{1}{3\sqrt{N}} \frac{(2\beta^3 -1)}{(2\beta^2 -1)^{3/2}}$$

$$C_4 = \frac{1}{4N} \frac{(2\beta^4 -1)}{(2\beta^2 -1)^2}$$

$$C_6 = \frac{1}{18N} \frac{(2\beta^3 -1)^2}{(2\beta^2 -1)^3} \qquad \text{(III.23)}$$

Appendix A

DERIVATION OF EQUATION II.8

Let

$$f_1 = \frac{Y_b - N + 1}{N} \; ; \quad f_2 = \frac{Y_b - \frac{N-1}{2}}{N^2}$$

then (referring to page 278 for definition of R_1 and R_2)

$$\left(\frac{R_1}{R_o}\right)^4 = \frac{-\ln P_D}{f_1} \tag{A1}$$

$$\ln P_D = -f_1 \left(\frac{R_2}{R_o}\right)^4 + f_2 \left(\frac{R_2}{R_o}\right)^8 \tag{A2}$$

so

$$\left(\frac{R_2}{R_o}\right)^4 = \frac{f_1}{2f_2}\left[1 - \sqrt{1 + \frac{4f_2 \ln P_D}{f_1^{\,2}}}\right] \tag{A3}$$

then, assuming $\frac{4f_2 \ln P_D}{f_1^{\,2}} << 1$,

$$\frac{R_2}{R_1} \approx 1 - \frac{f_2}{4f_1^{\,2}} \ln P_D = 1 - \mathcal{E}(n,N) \ln P_D \tag{A4}$$

where

$$\mathcal{E}(n,N) = \frac{Y_b - \frac{N-1}{2}}{4(Y_b - N + 1)^2} \tag{A5}$$

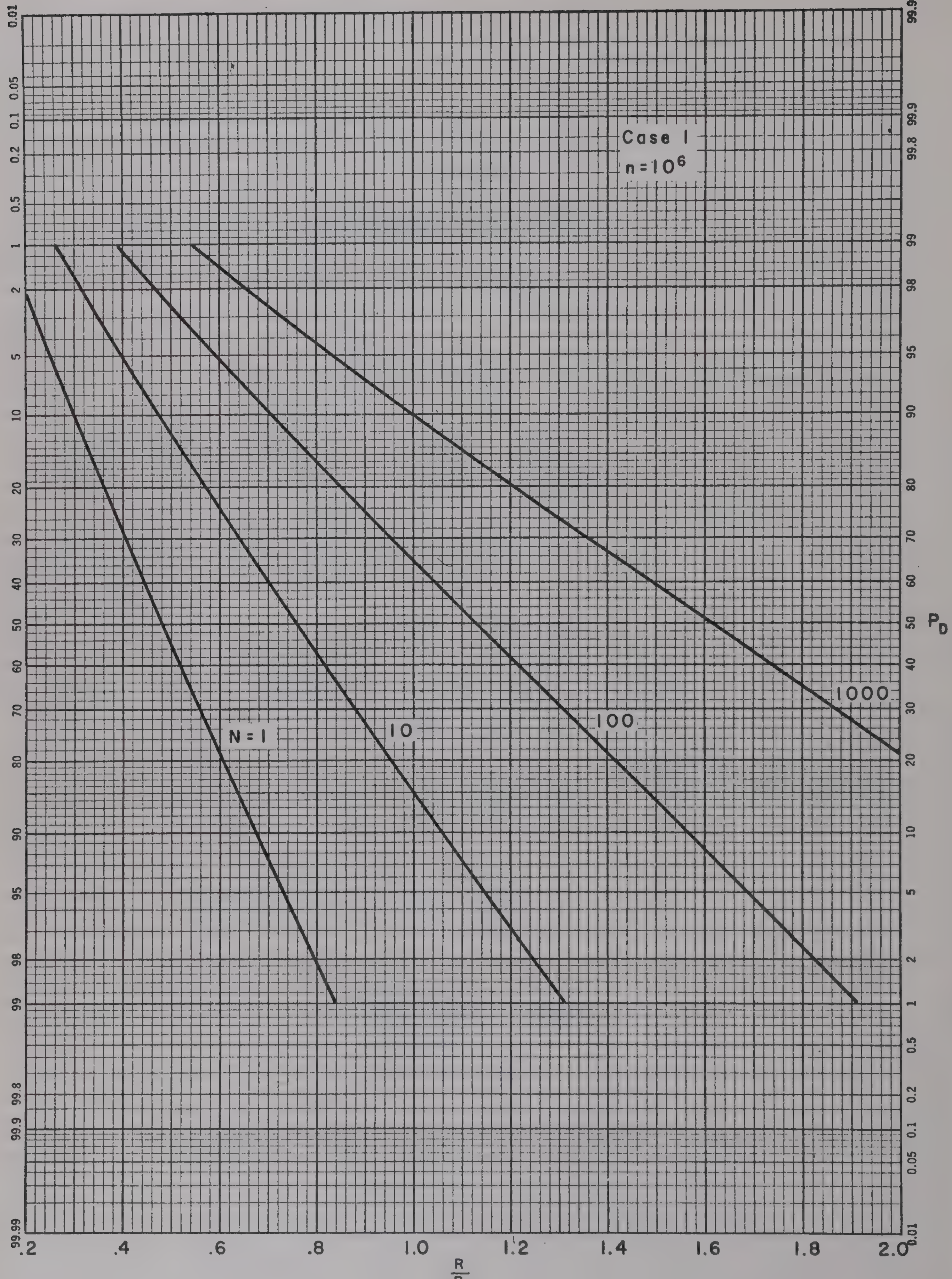
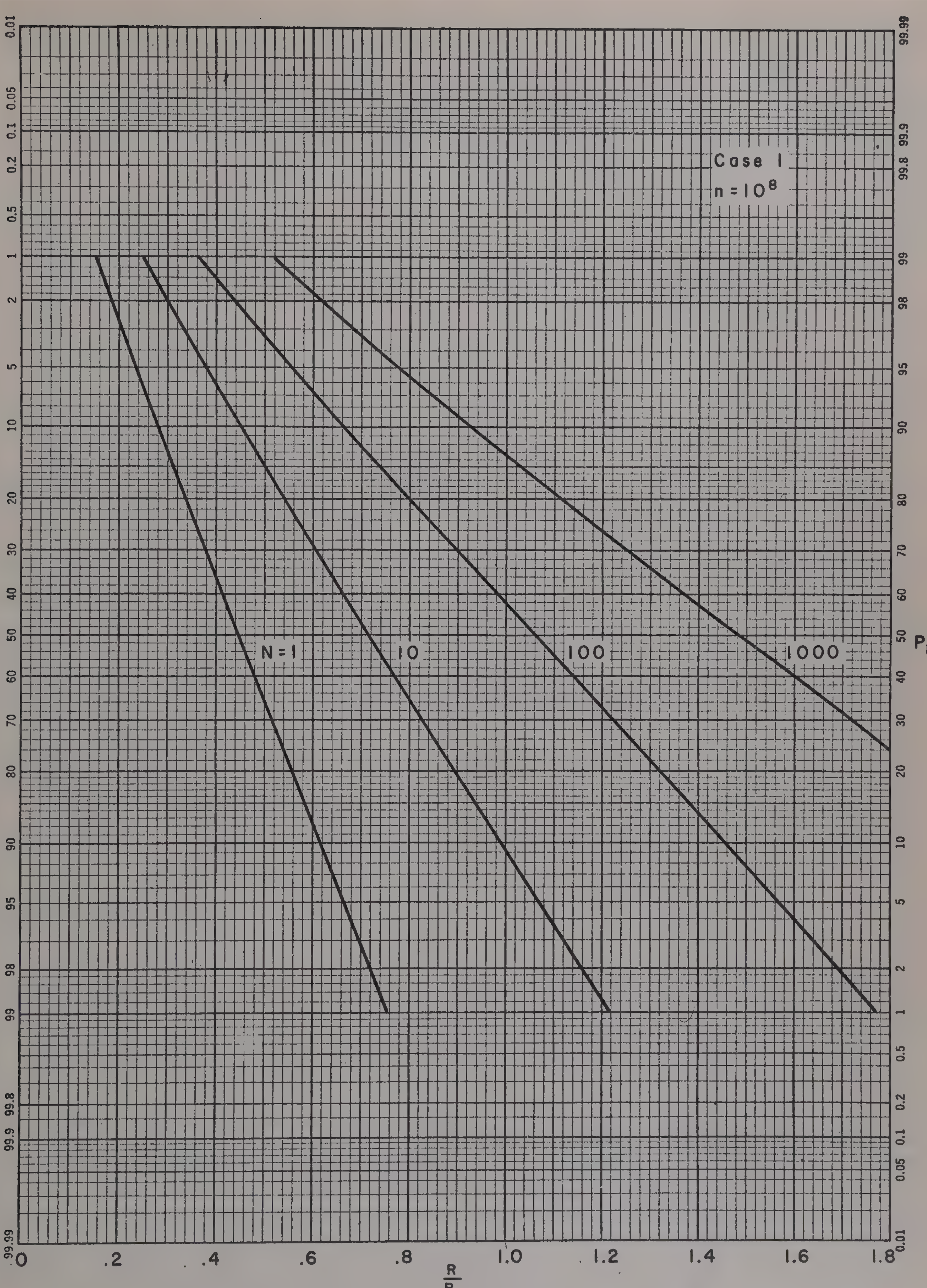


Fig. I



Case I
n = 10^8
N = 1
10
100
1000
P_D
0.01
0.05
0.1
0.2
0.5
1
2
5
10
20
30
40
50
60
70
80
90
95
98
99
99.8
99.9
99.99
0
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
R/R_0

Fig. 2

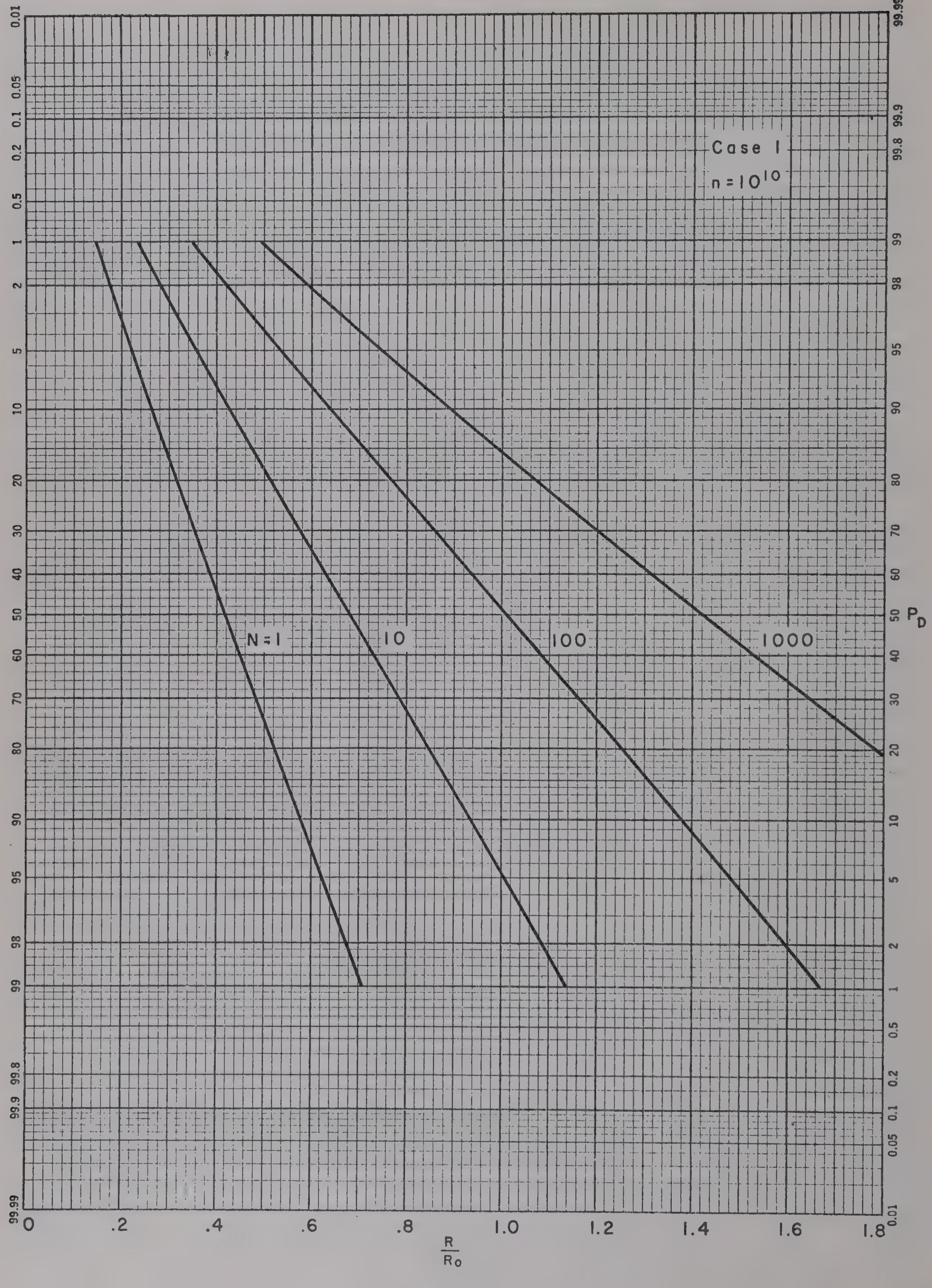


Fig. 3

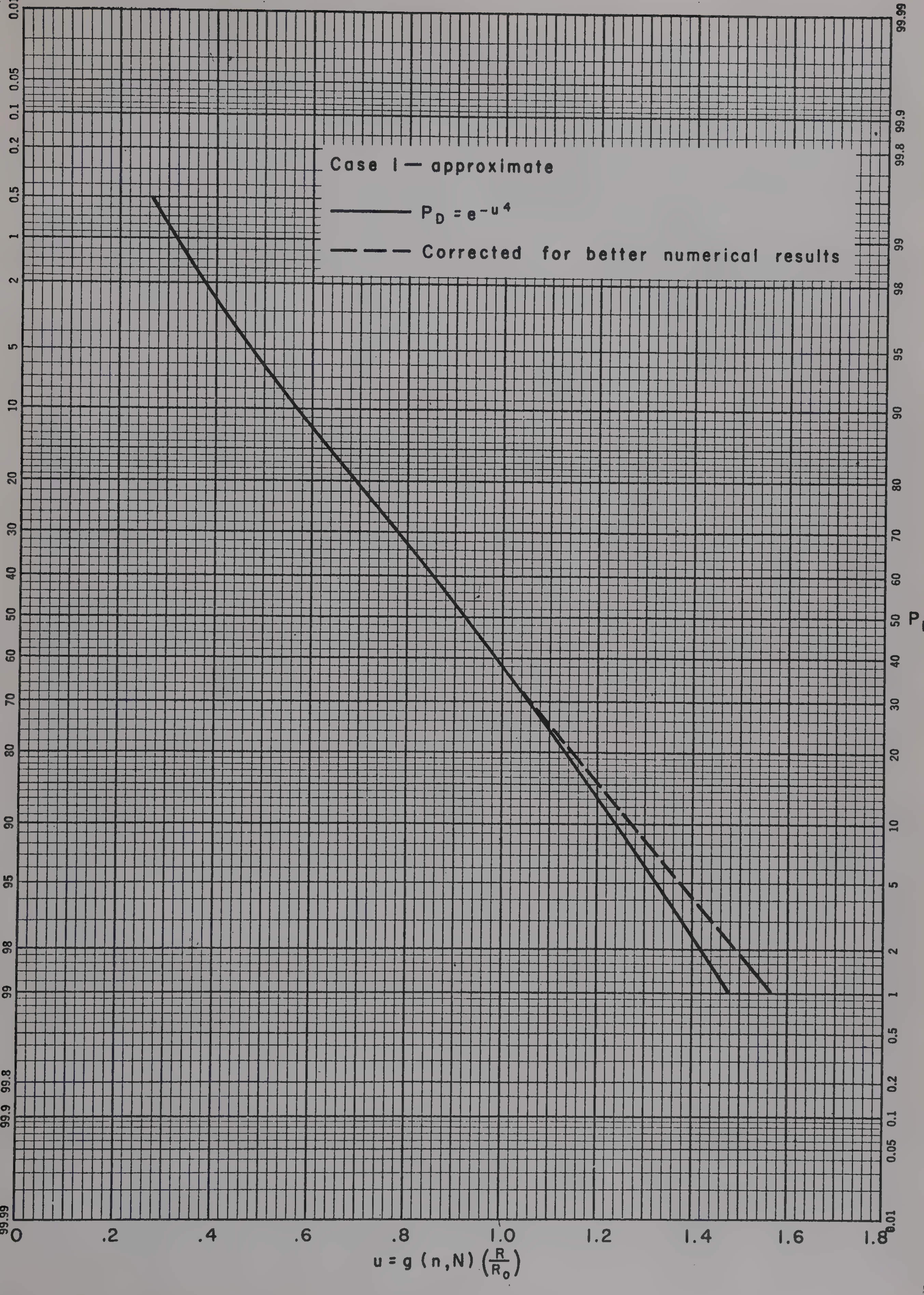
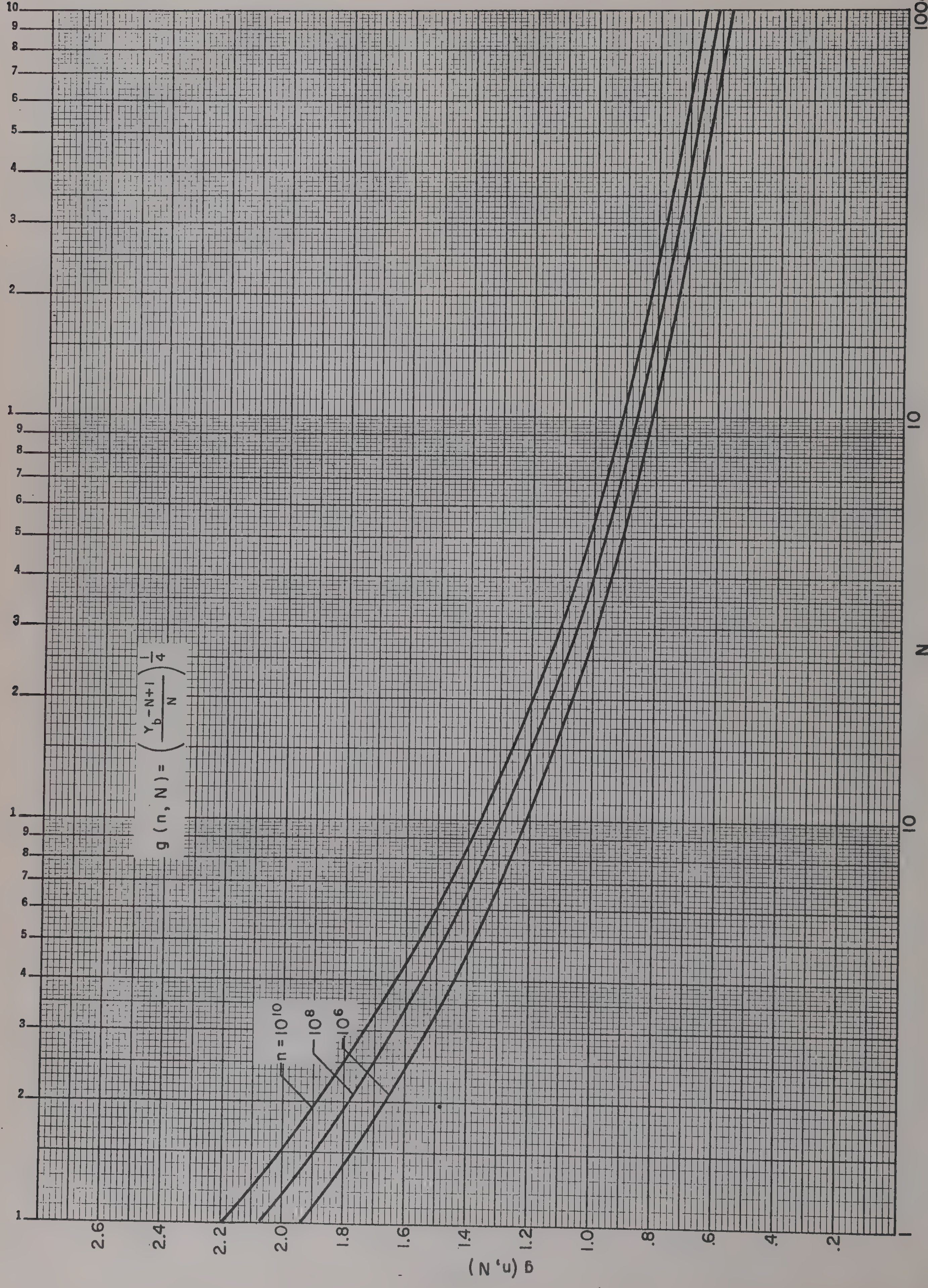
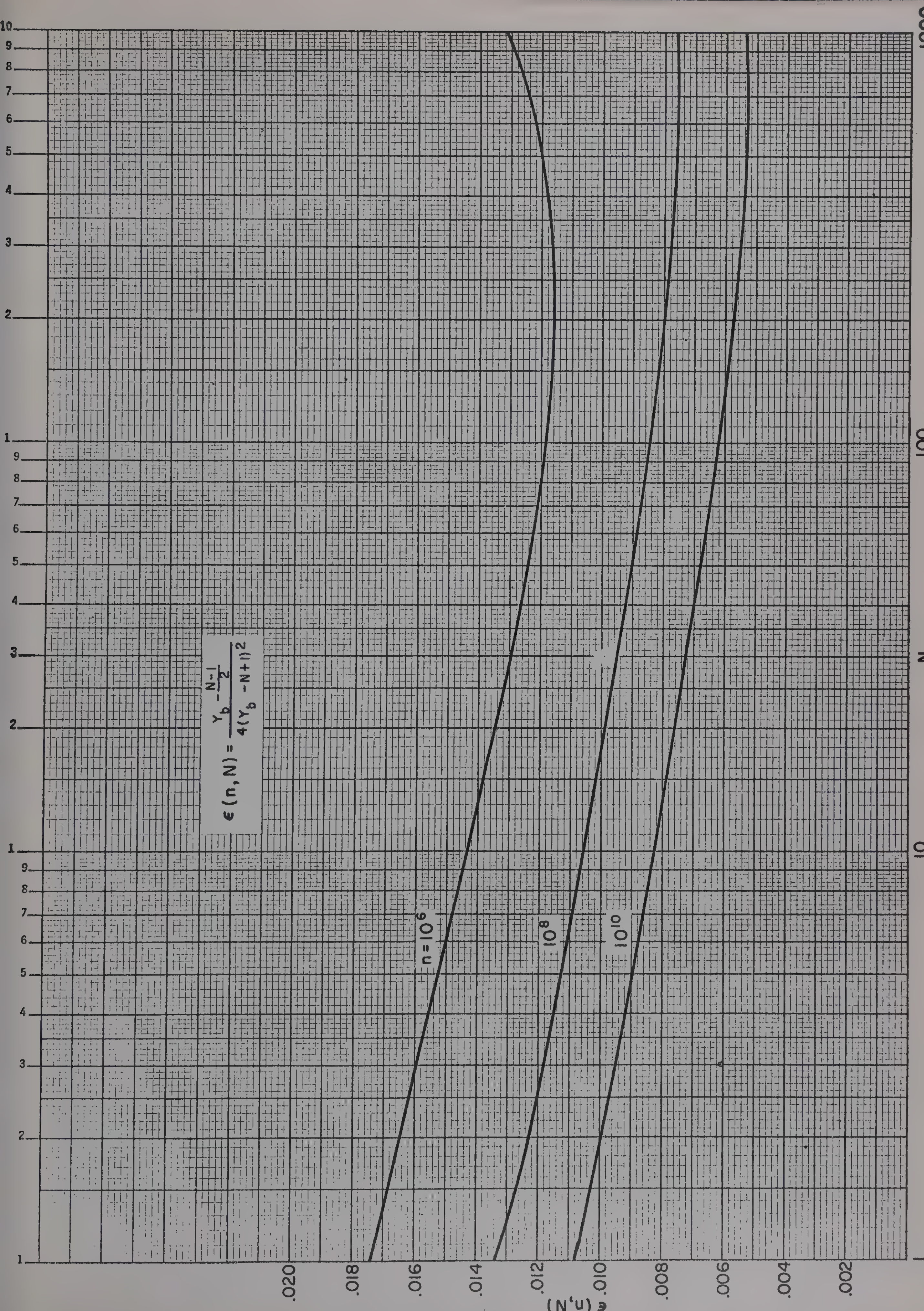

Case I — approximate
$P_D = e^{-u^4}$
Corrected for better numerical results
P_D
$u = g(n,N)\left(\frac{R}{R_0}\right)$
0
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
0.01
0.05
0.1
0.2
0.5
1
2
5
10
20
30
40
50
60
70
80
90
95
98
99
99.8
99.9
99.99

Fig. 4

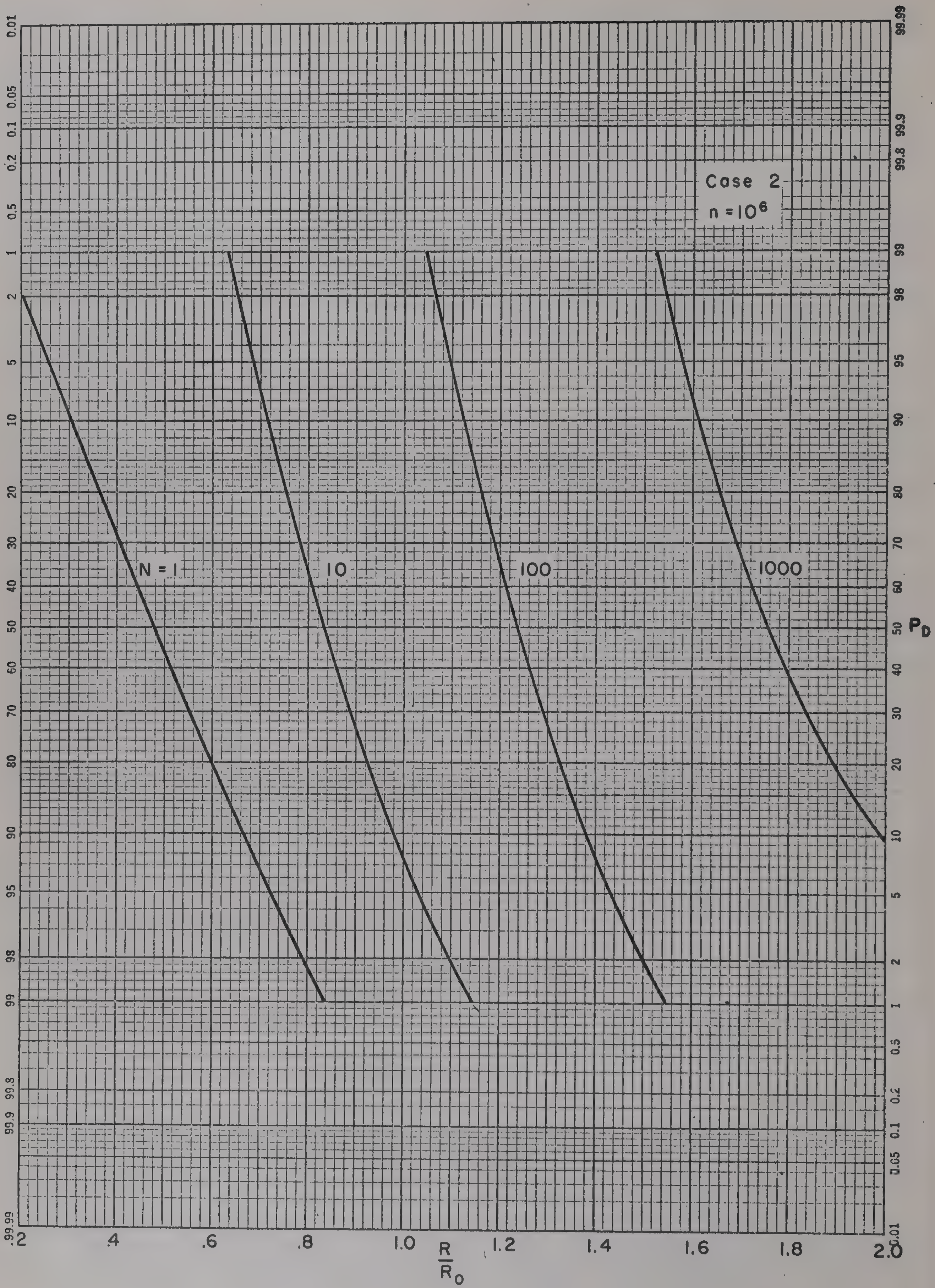

g (n, N) = ((Y_b − N + 1) / N)^(1/4)
n = 10^10
10^8
10^6
g (n, N)
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
.8
.6
.4
.2
1
10
1000

N
Fig. 5



$\epsilon(n,N) = \dfrac{Y_b - \frac{N-1}{2}}{4(Y_b - N+1)^2}$
n=10^6
10^8
10^{10}
ϵ (n,N)
.020
.018
.016
.014
.012
.010
.008
.006
.004
.002
1
10
100
1000
N

Fig. 5



Case 2
n = 10^6
N = 1
10
100
1000
P_D
R/R_0
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
2.0
0.01
0.05
0.1
0.2
0.5
1
2
5
10
20
30
40
50
60
70
80
90
95
98
99
99.8
99.9
99.99

Fig. 7

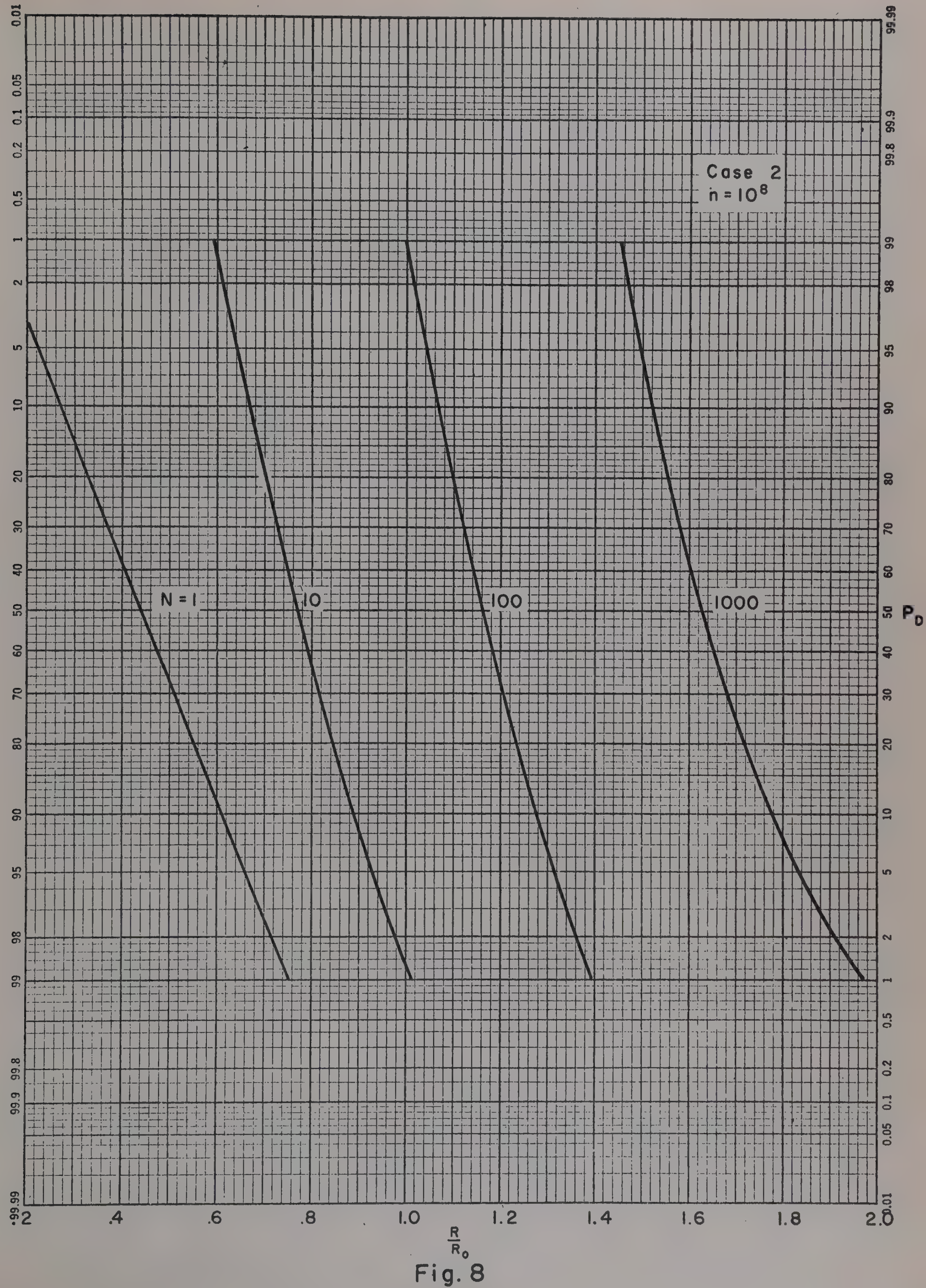


Fig. 8

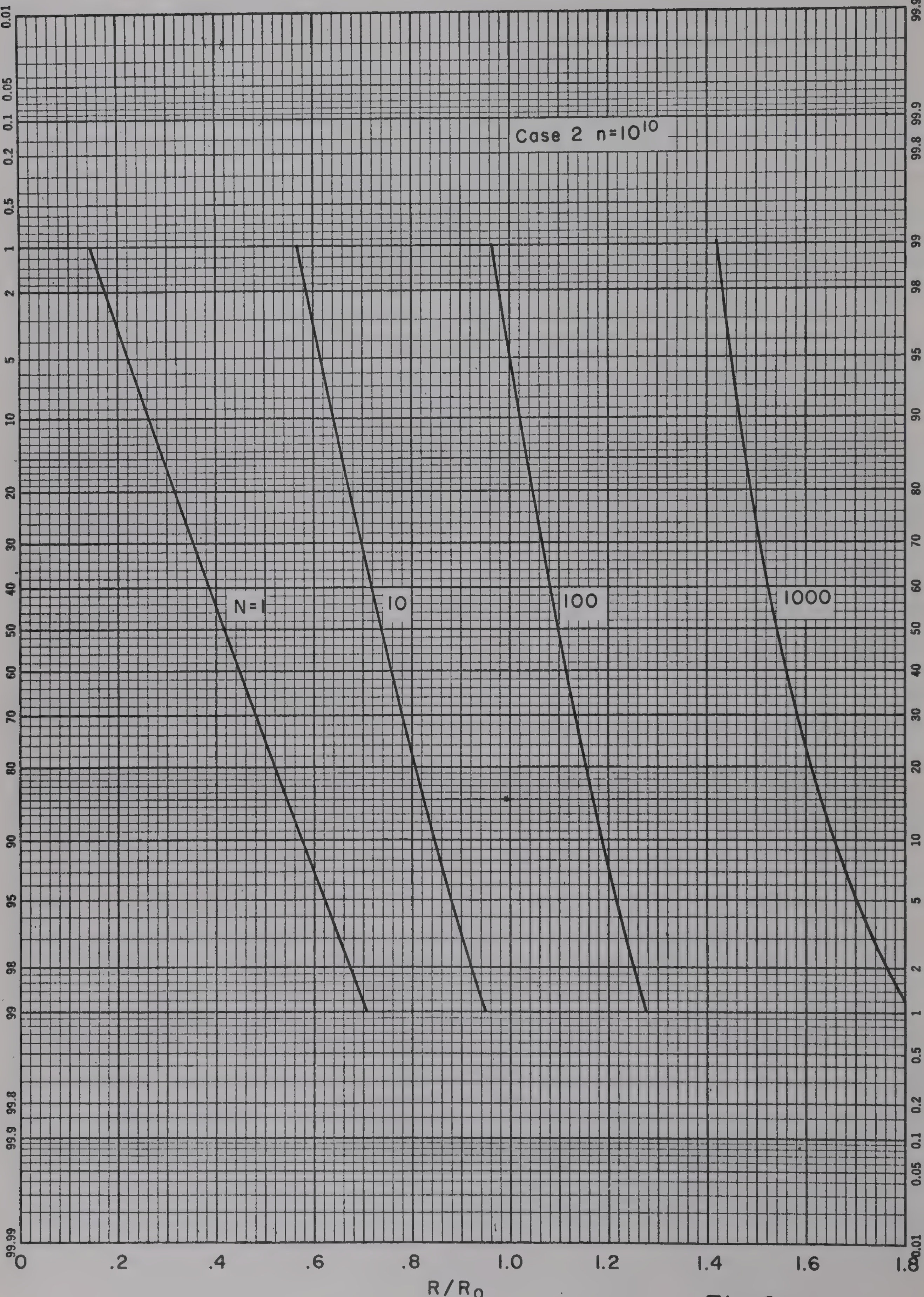
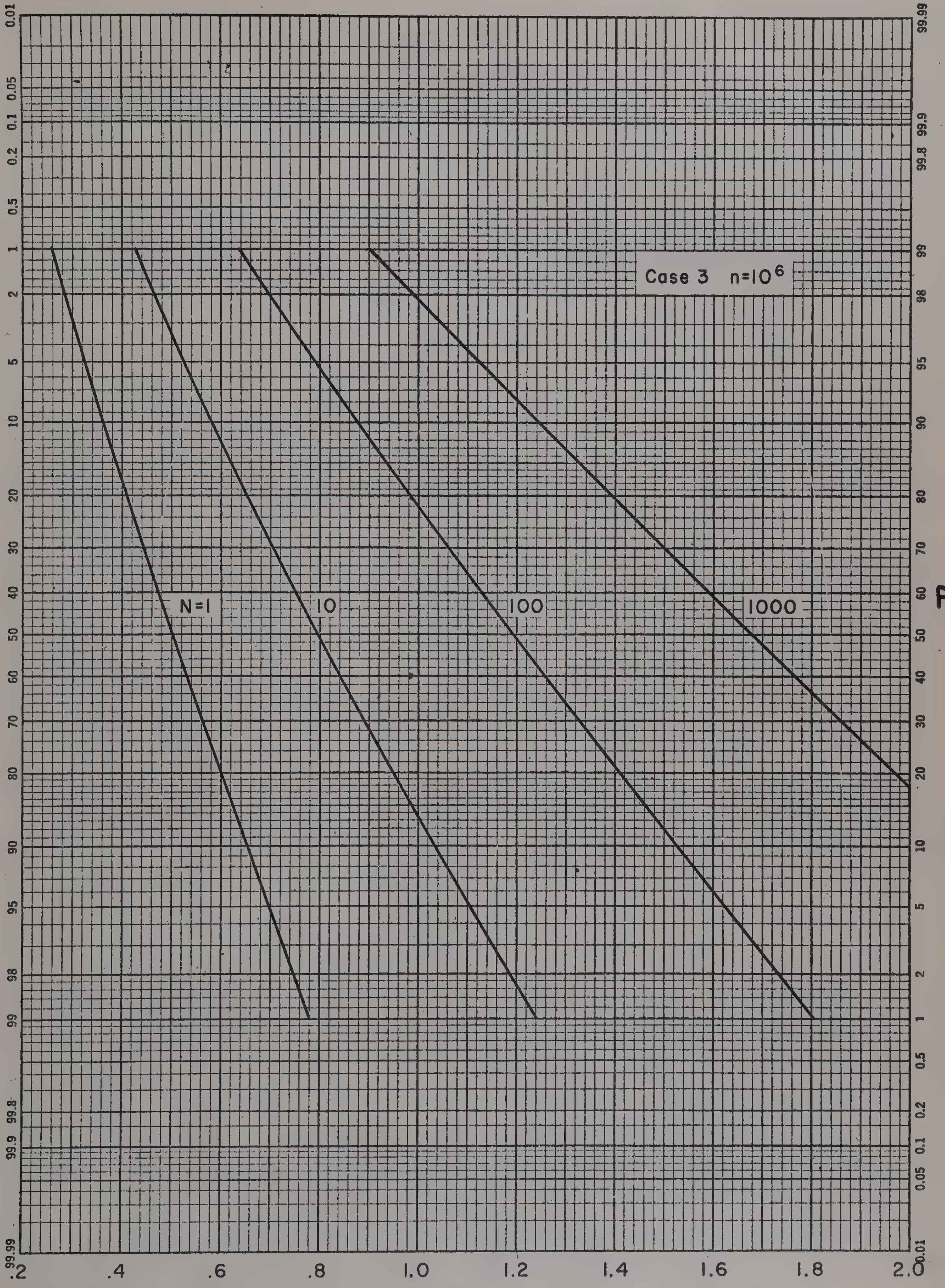


Fig. 9



0.01
0.05
0.1
0.2
0.5
1
2
5
10
20
30
40
50
60
70
80
90
95
98
99
99.8
99.9
99.99
99.99
99.9
99.8
99
98
95
90
80
70
60
50
40
30
20
10
5
2
1
0.5
0.2
0.1
0.05
0.01
Case 3 n=10^6
N=1
10
100
1000
P_D
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
2.0
R/R0

Fig. 10

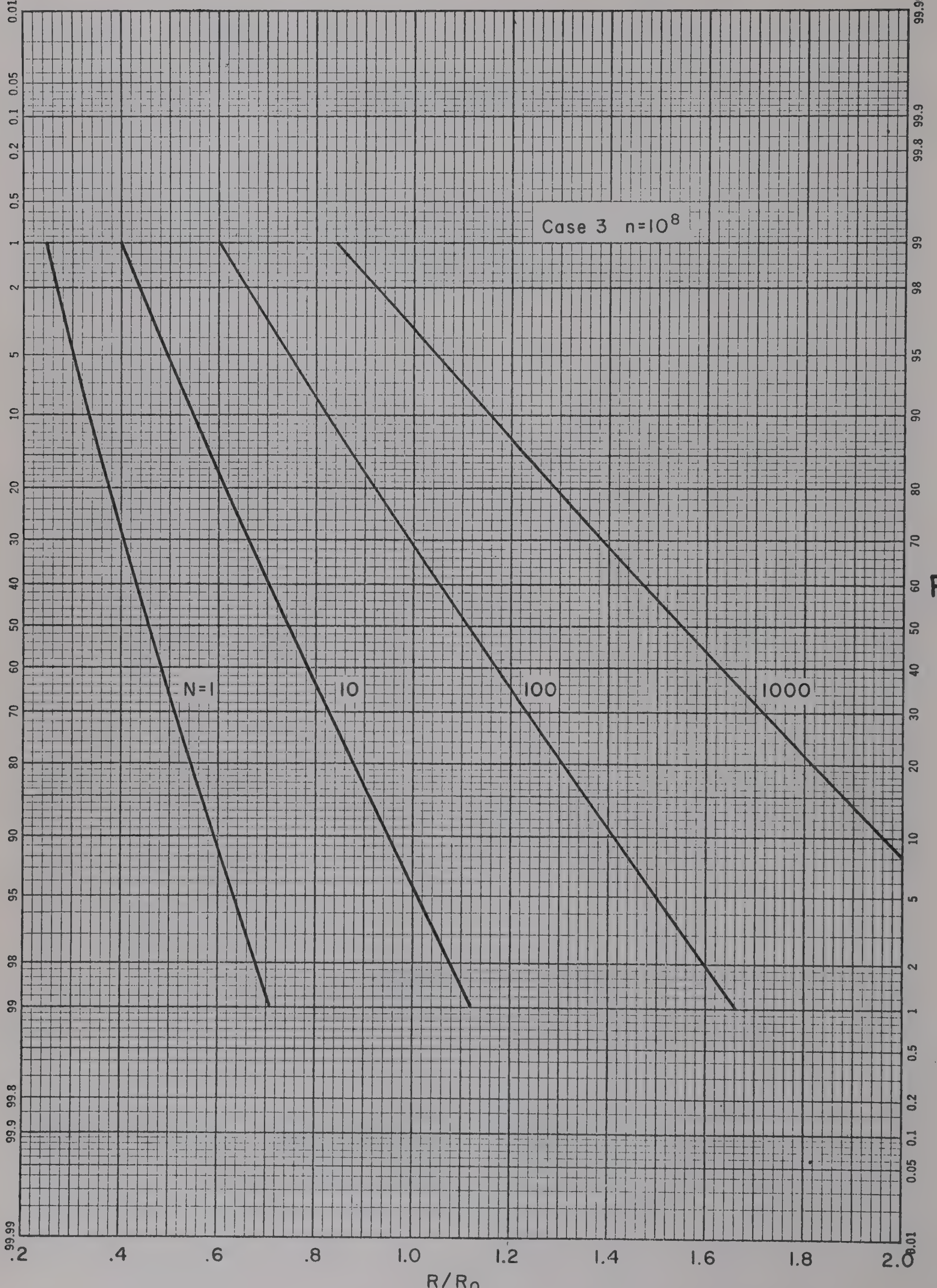


Fig. 11

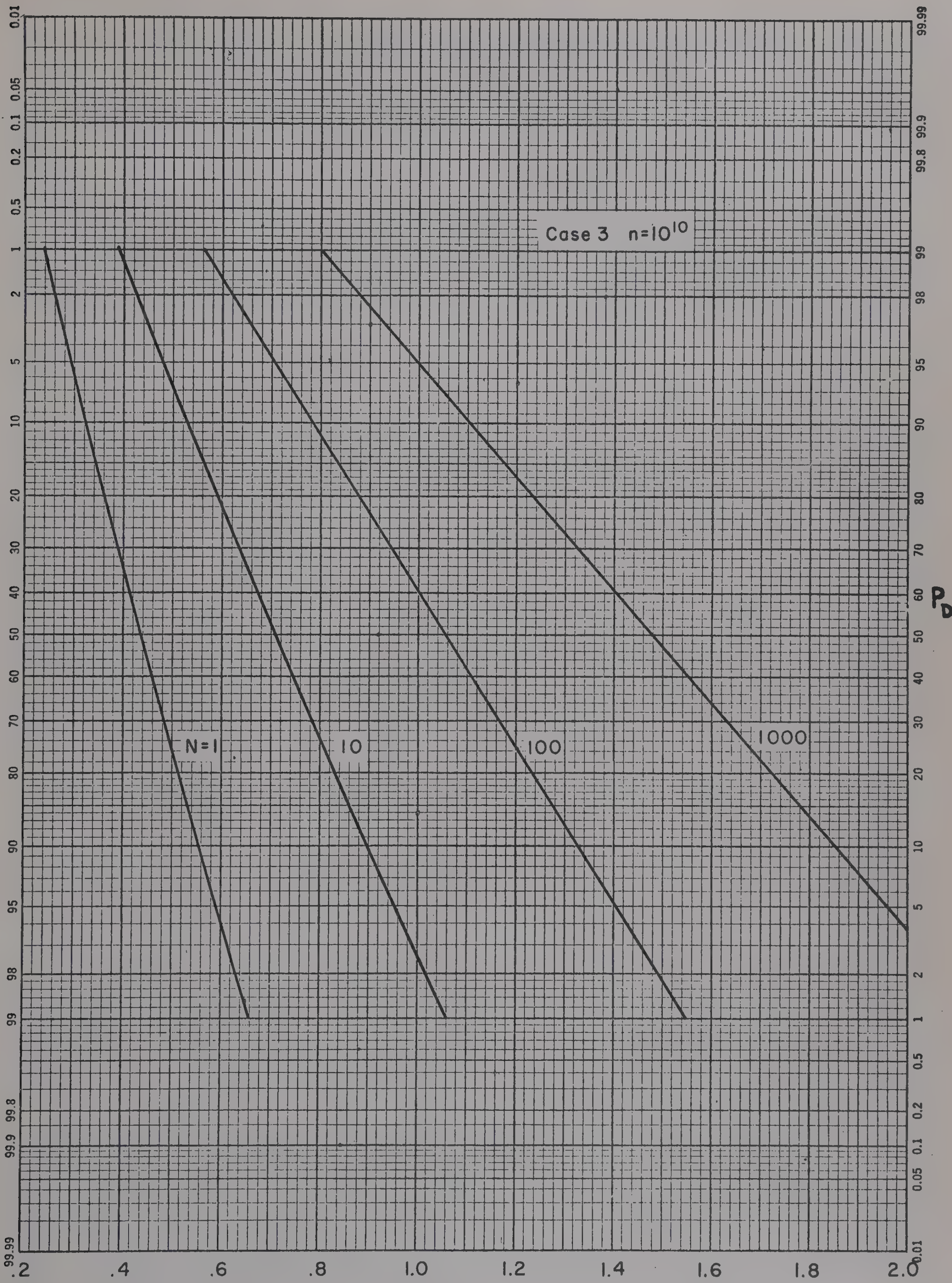
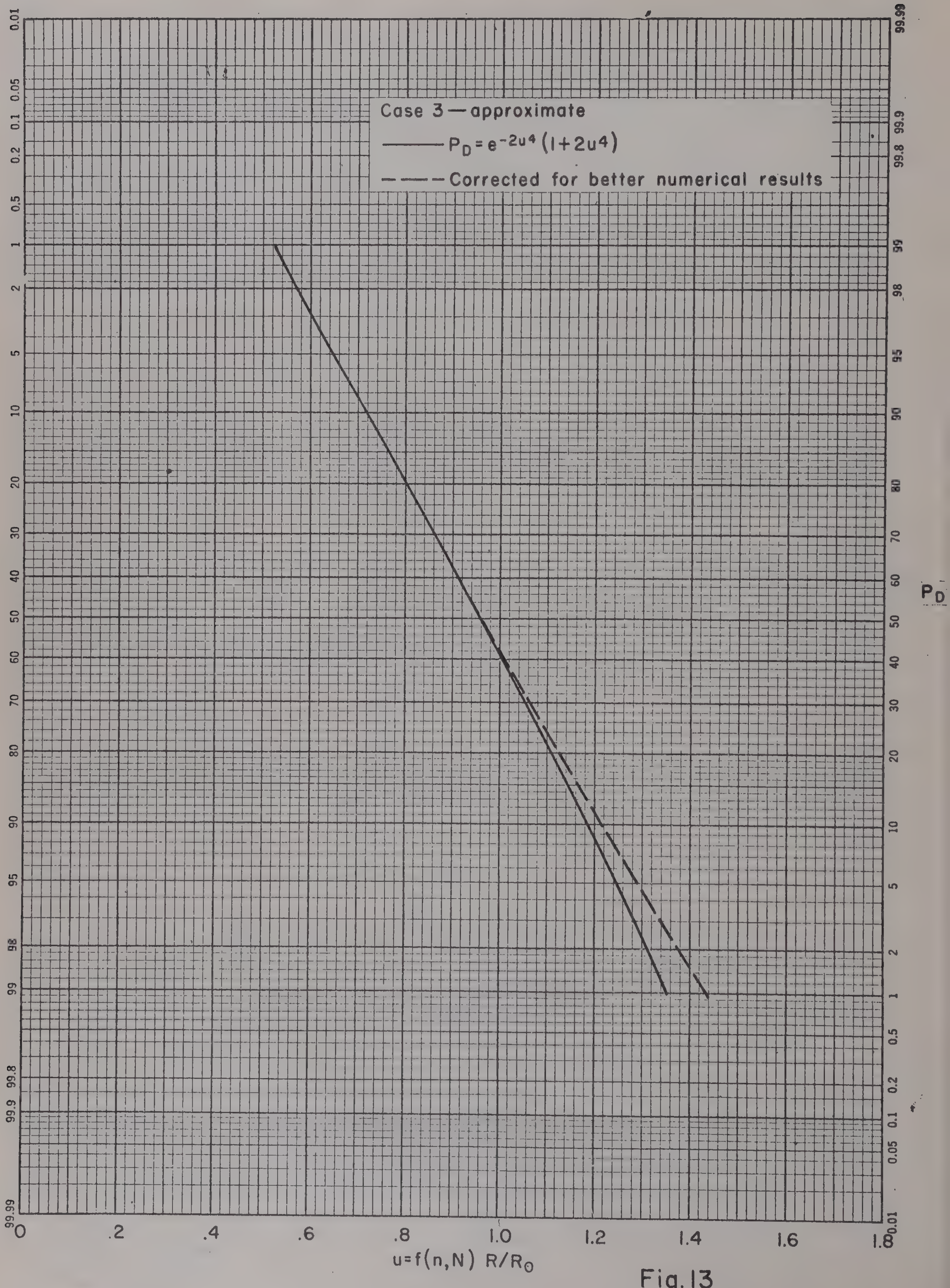


Fig. 12



Case 3 — approximate
$P_D = e^{-2u^4}(1+2u^4)$
Corrected for better numerical results
P_D
$u = f(n,N)\ R/R_0$
0
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
0.01
0.05
0.1
0.2
0.5
1
2
5
10
20
30
40
50
60
70
80
90
95
98
99
99.8
99.9
99.99

Fig. 13

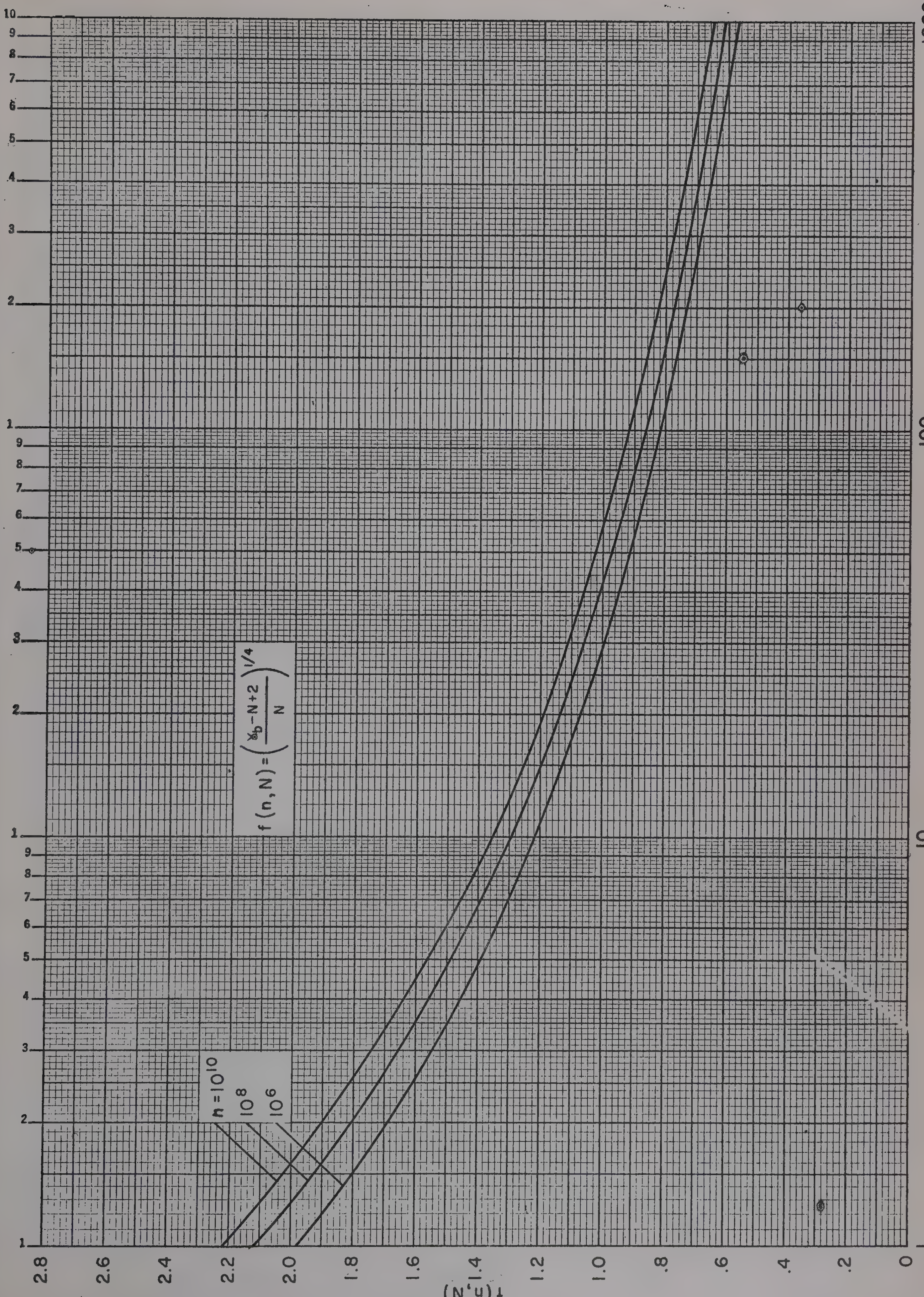

f(n, N) = ((X_b − N + 2) / N)^{1/4}
n = 10^10
10^8
10^6
f(n,N)
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
.8
.6
.4
.2
0
1
10
100
1000

Fig. 14

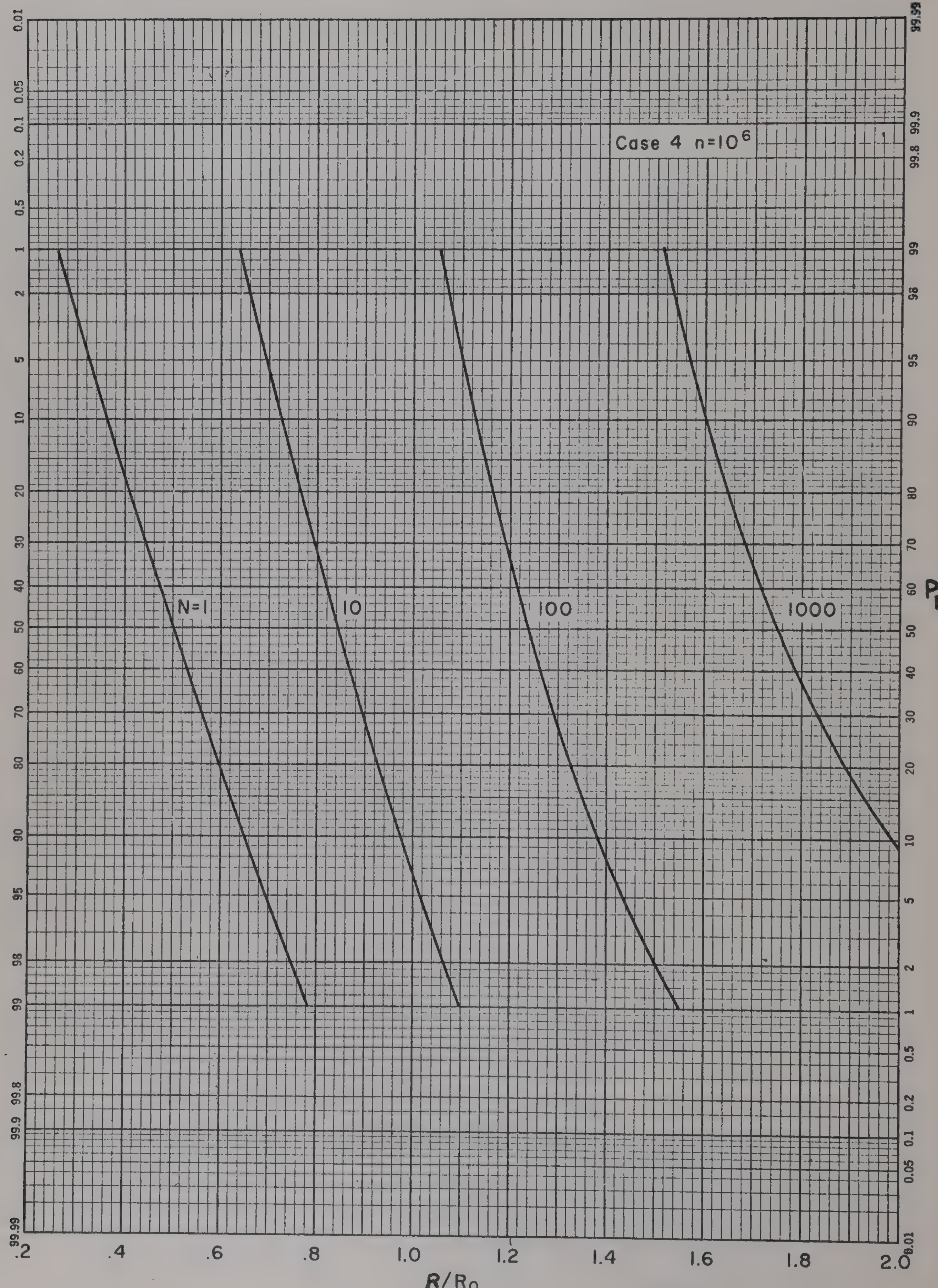


Fig. 15

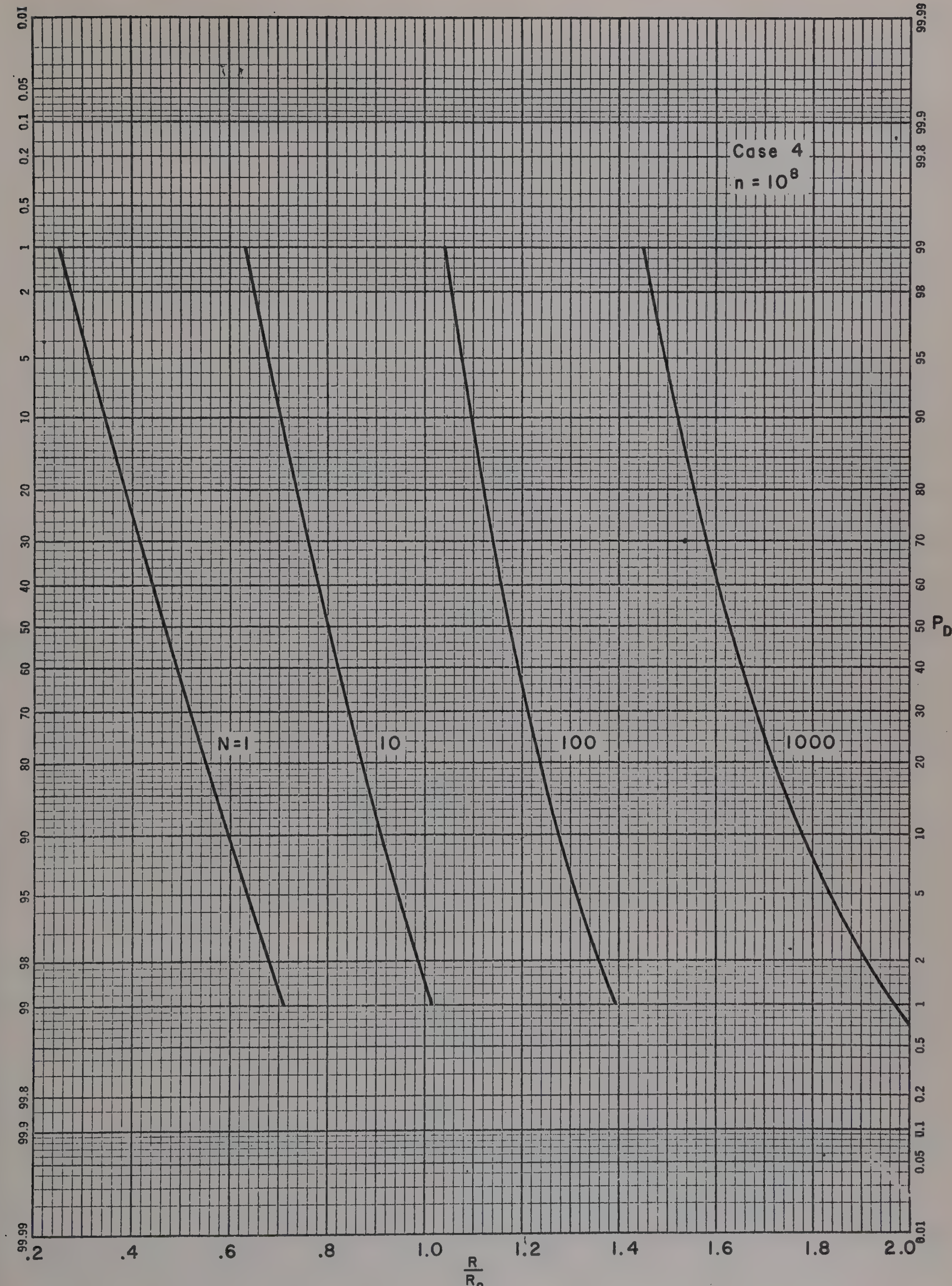


Fig. 16

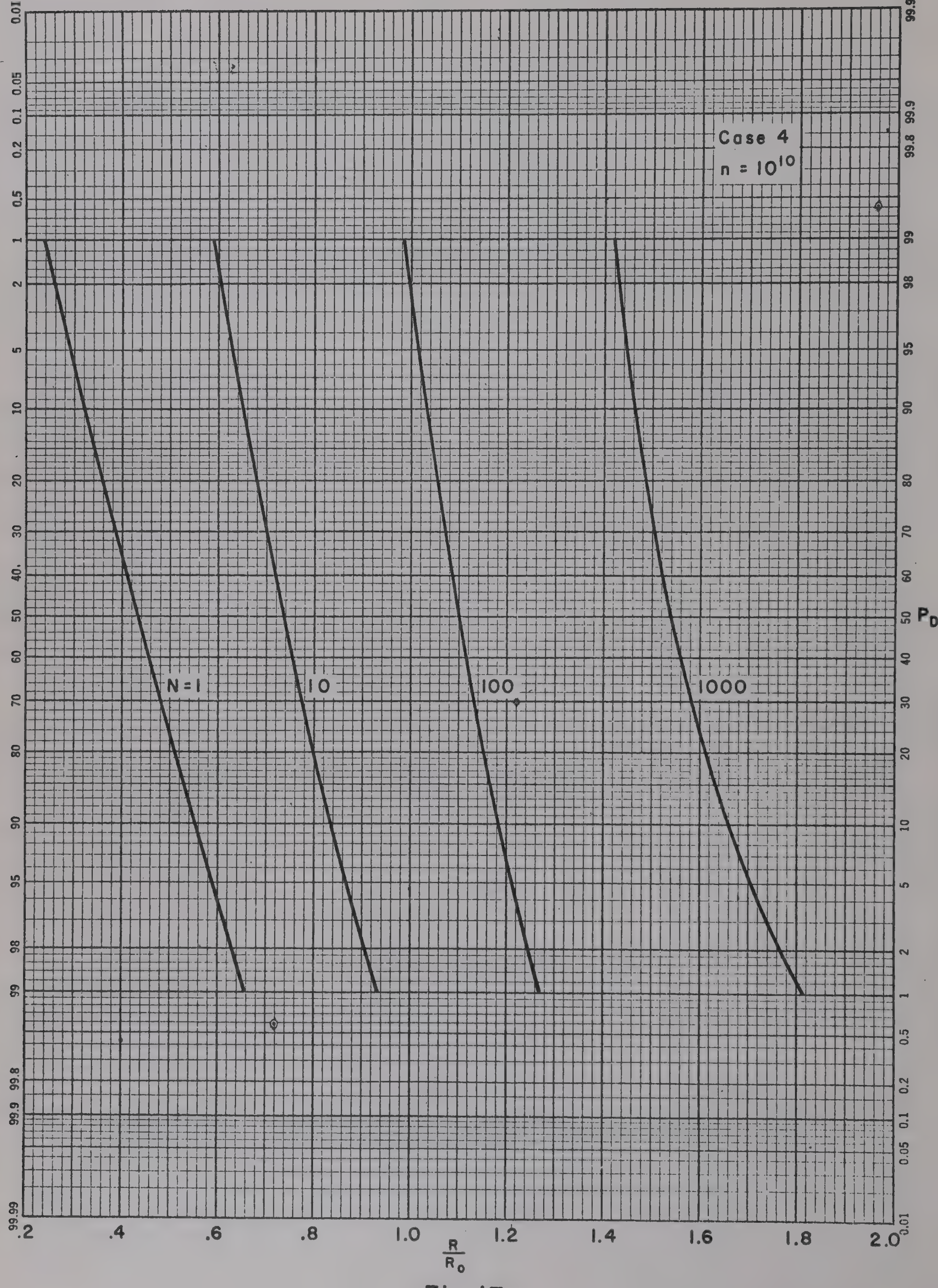


Fig.17

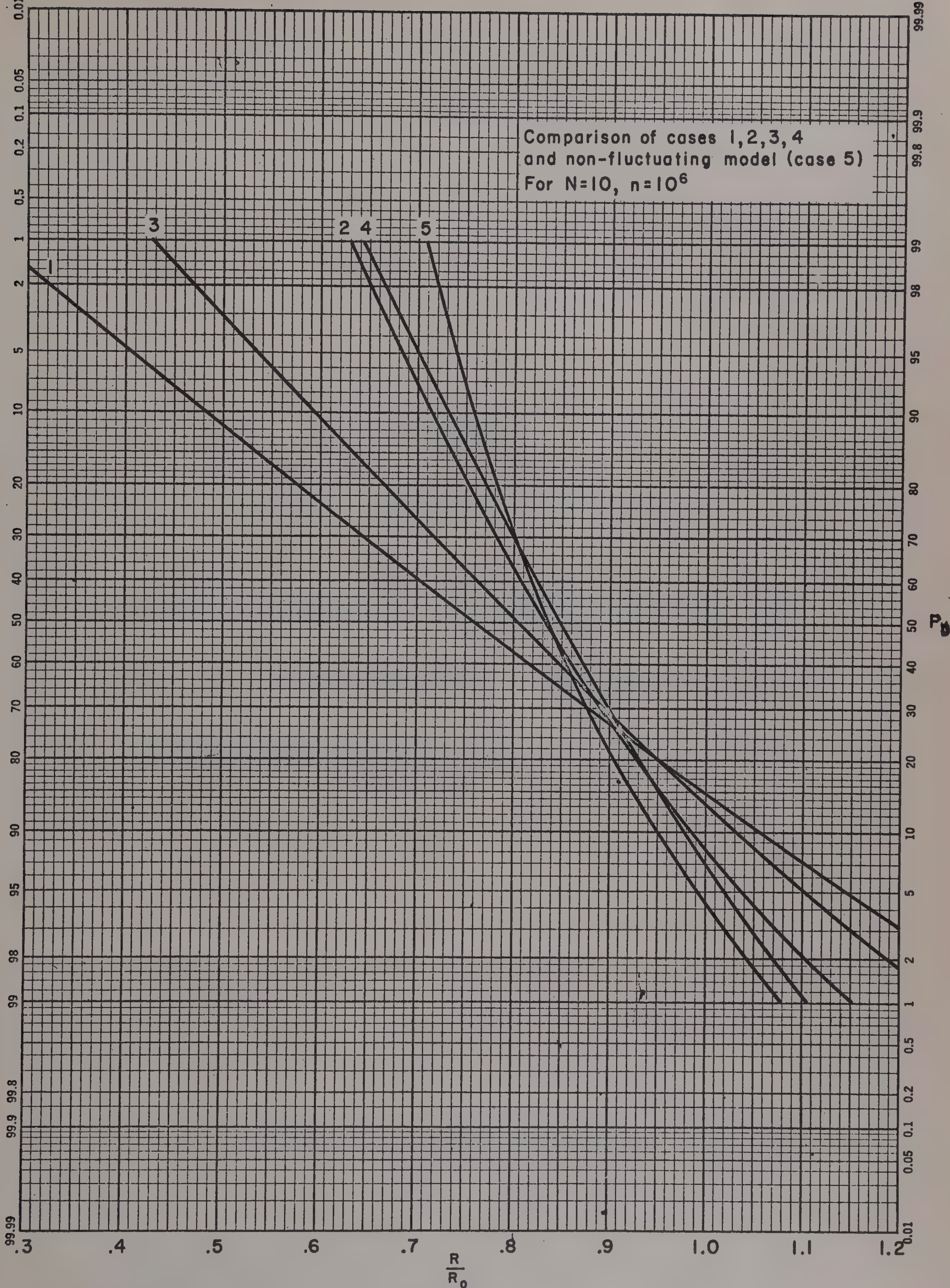

Comparison of cases 1,2,3,4
and non-fluctuating model (case 5)
For N=10, n=10^6
1
2
3
4
5
P_D
$\frac{R}{R_0}$

Fig. 18

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3. Pearson, K., Tables of the Incomplete Gamma Function, Cambridge University Press, 1946.

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